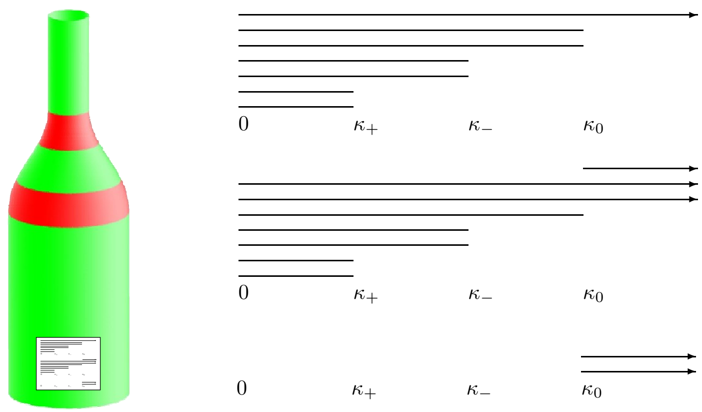
Persistence Barcodes for Shapes



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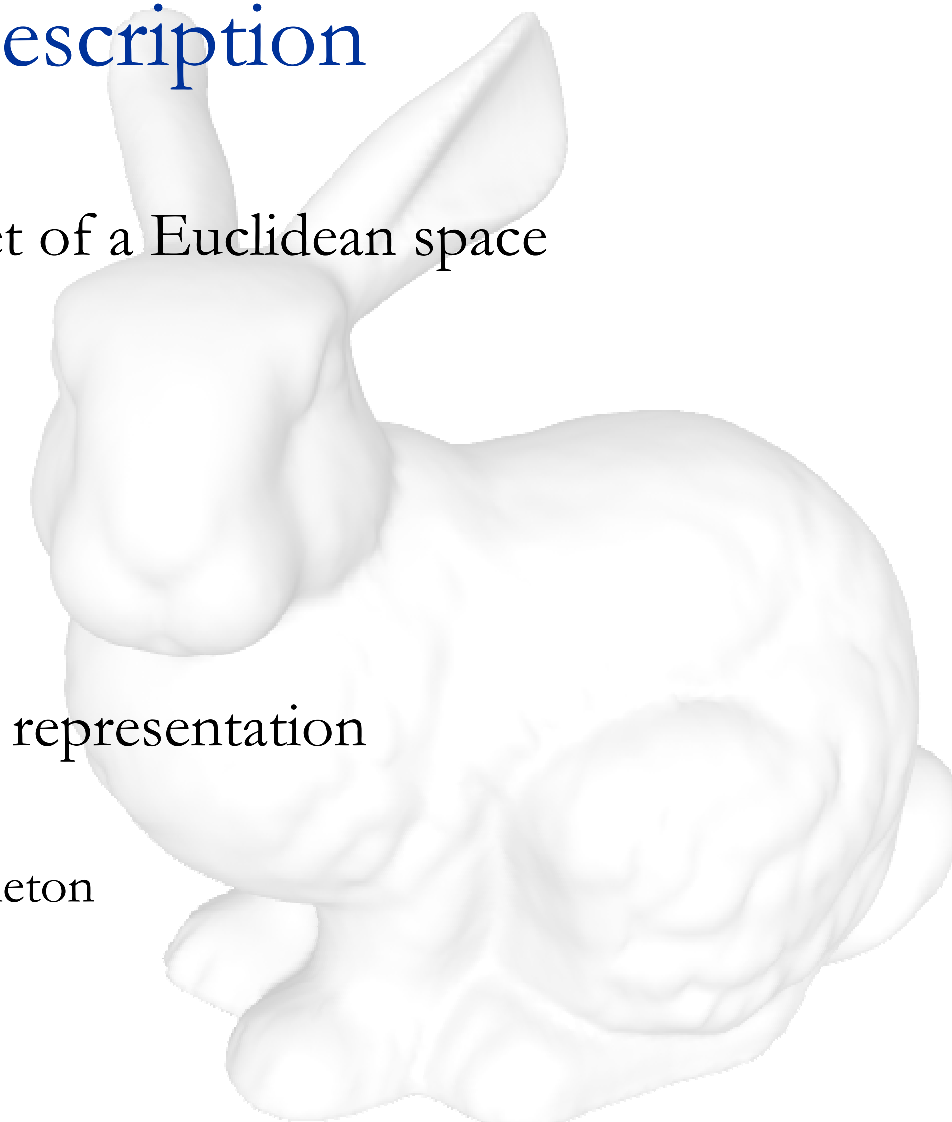
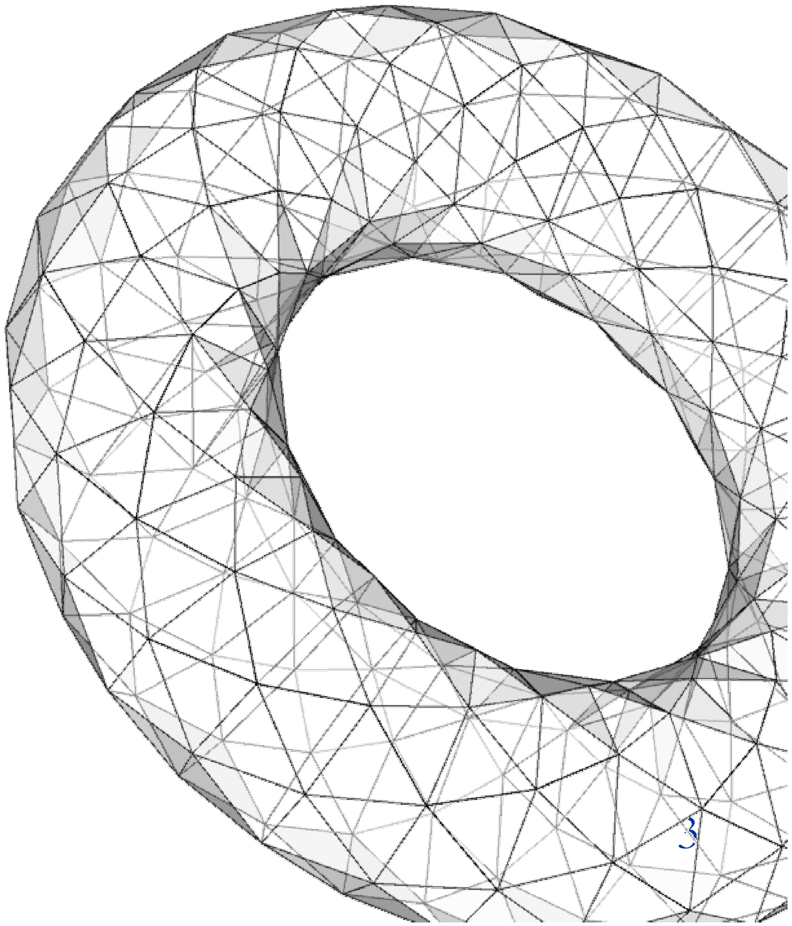
Shape Description



- A shape is any closed subset of a Euclidean space
- Questions:
  - Feature identification
  - Matching
  - Classification
  - Parameterization of a family
- Shape descriptor: compact representation
  - Moment invariants
  - Medial axis transform or skeleton
  - Fourier parametrization
  - Shape distributions
  - Multiresolution Reeb graphs

Geometry versus Topology

- Euclidean geometry
  - What does the shape look like?
  - Local
  - Quantitative
  - Low-level
- Topology
  - How is a shape connected?
  - Global
  - Qualitative
  - High-level



Homology

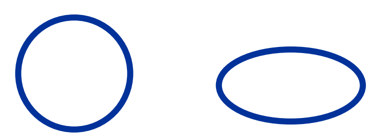
- The Betti numbers count topological attributes:
  - β_0: # components
  - β_1: # tunnels or loops
  - β_2: # voids

F $\beta_1 = 0$ A $\beta_1 = 1$ B $\beta_1 = 2$

- Problem:
  - Cannot detect sharp features

U $\beta_1 = 0$ V $\beta_1 = 0$

  - Cannot detect soft features



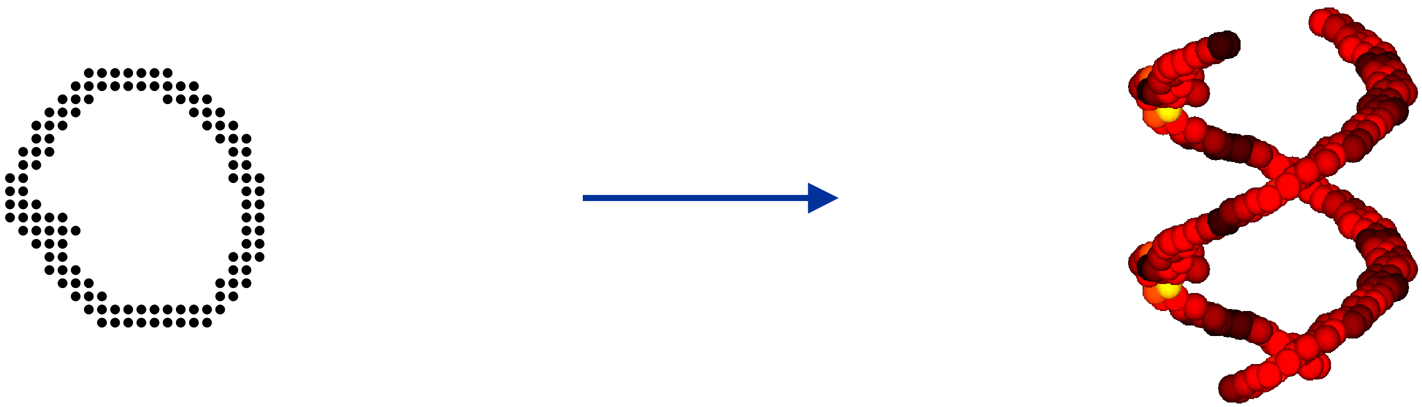
$\beta_1 = 1$ $\beta_1 = 1$

Approach

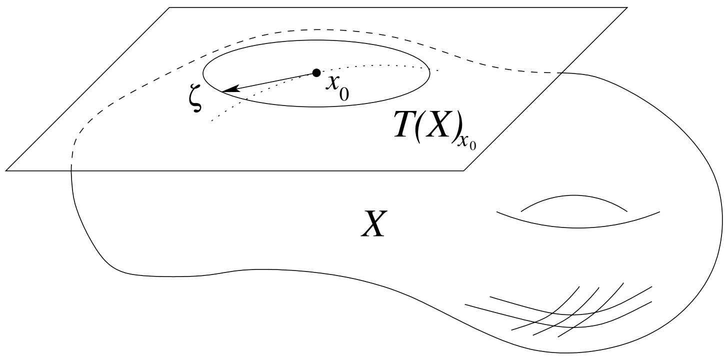
- Geometry: *differentiating* power
- Topology: *classifying* power
- Idea: combine the two
  - Apply topology to a *derived* space
  - Enrich derived space with geometric information
- Tools
  - Tangent Complex
  - Filtered by curvature
  - Persistent Homology
- Contribution
  - Barcode shape descriptor
  - Metric over the space of barcodes
  - Uniform framework

Plan

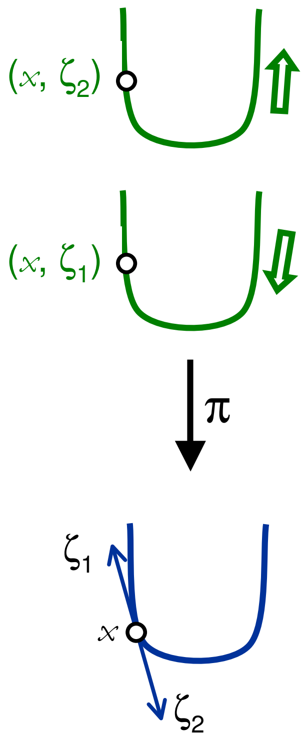
- Today: *Theory*
  - Definitions
  - Analysis of mathematical shapes
- Tomorrow @ 3:40: *Practice*
  - A Barcode Shape Descriptor for Curve Point Cloud Data by Anne Collins



Tangent Complex



Curve Tangent Complex

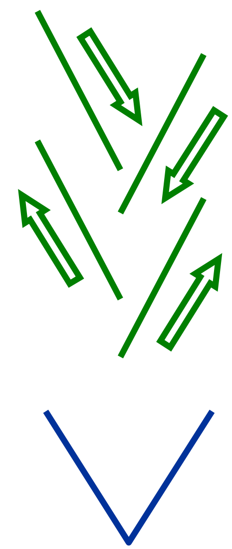


$T(X)$ has two components: $\beta_0(T(X)) = 2$

There are two points in its fiber $\pi^{-1}(x)$

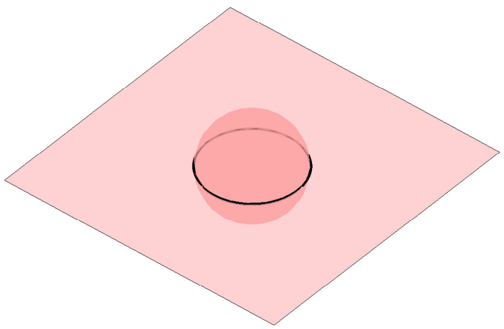
Every point x on a smooth curve X has two tangent directions.

A corner point has four tangent directions: $\beta_0(T(X)) = 4$

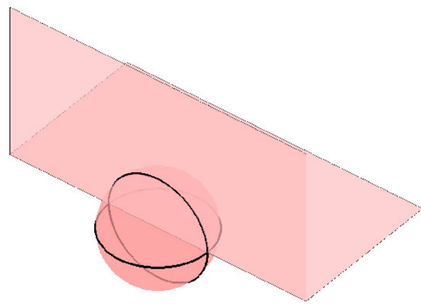


Surface Fibers

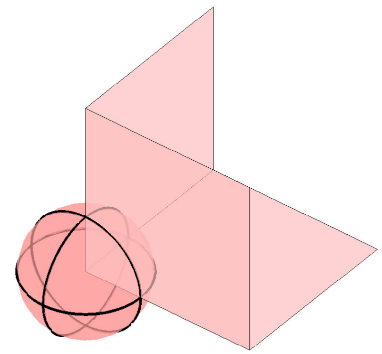
- The fiber for a point on a surface is a set of circles that intersect pairwise



Smooth: 1 circle
or $\mathbb{S}^1$

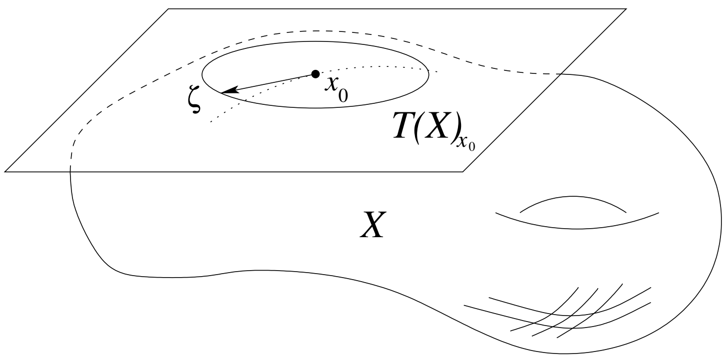


Edge: 2 circles
or $(\mathbb{S}^1)^3$



Cone: 3 circles
or $(\mathbb{S}^1)^7$

Tangent Complex



- Let $X \subseteq \mathbb{R}^n$ be a space.
- Tangent ζ at point $x \in X$
- $(x, \zeta) \in T^0(X) \subseteq X \times \mathbb{S}^{n-1}$

$$T^0(X) = \left\{ (x, \zeta) \mid \lim_{t \to 0} \frac{\mathrm{d}(x + t\zeta, X)}{t} = 0 \right\}$$

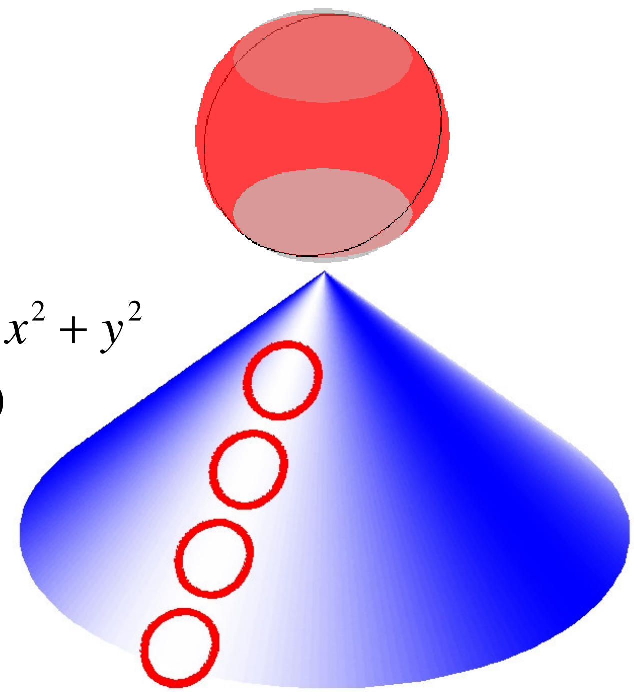
- The tangent complex $T(X)$ of X is the closure of T^0
- The projection $\pi : T(X) \to X$ projects (x, ζ) to its basepoint x
- Basepoint $x \in X$ has fiber $T(X)_x = \pi^{-1}(x) \in \mathbb{S}^{n-1}$

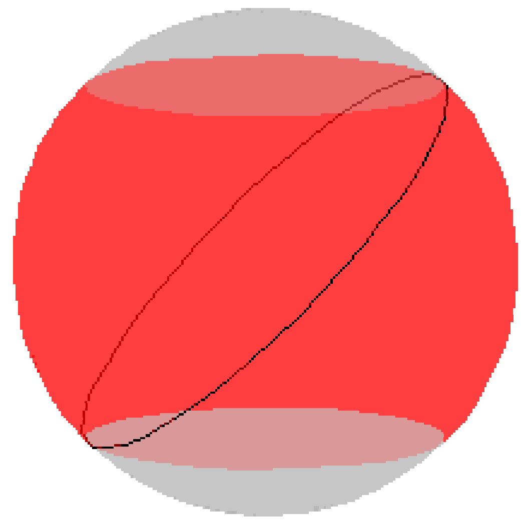
Detecting a Cone Point

- A flat disc $\mathbb{D}^2$ has tangent complex $\mathbb{D}^2 \times \mathbb{S}^1$
- What about a cone?

$z^2 = x^2 + y^2$

$z \leq 0$





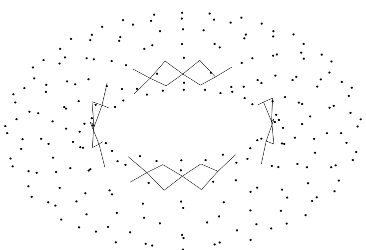
Annulus $\cong \mathbb{S}^1$

Persistence

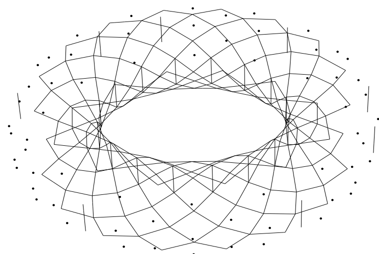
$$
\begin{array}{ccccccc}
\partial_3 \big\downarrow & & \partial_3 \big\downarrow & & \partial_3 \big\downarrow & & \\
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \cdots \\
\partial_2 \big\downarrow & & \partial_2 \big\downarrow & & \partial_2 \big\downarrow & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \cdots \\
\partial_1 \big\downarrow & & \partial_1 \big\downarrow & & \partial_1 \big\downarrow & & \\
C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \cdots
\end{array}
$$

Incremental Construction

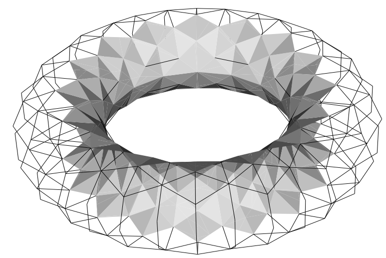
- Suppose we construct a space incrementally:



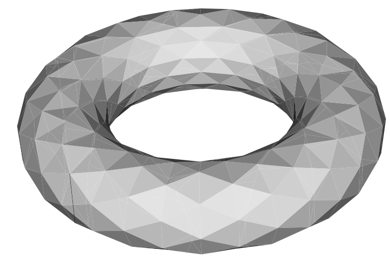
$\subseteq$



$\subseteq$



$\subseteq$



$\beta_0 = 220$	$\beta_0 = 73$	$\beta_0 = 1$	$\beta_0 = 1$
$\beta_1 = 8$	$\beta_1 = 161$	$\beta_1 = 177$	$\beta_1 = 2$
$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 0$	$\beta_2 = 1$

- As the space evolves, topological attributes are created and destroyed.

Filtrations

- A filtration of a complex K is a nested sequence of complexes
$$\emptyset = K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m = K$$
- K is a filtered complex
- Natural
  - Cěch-like complexes
  - Density measure
  - Manifolds equipped with Morse functions

- Problem: looking for features in this growing space

Persistent Homology

- ATMCS 2001
  - $H_k^{l,p} = Z_k^l / (B_k^{l+p} \cap Z_k^l)$ [ELZ02]
  - Topological attributes have lifetime of existence
  - Algorithm for $\mathbb{Z}_2$-homology, subcomplexes of triangulated $\mathbb{S}^3$
  - Proof via canonical bases
- But *why* does it work?
  - Why can we pair and get intervals?
  - What's the relationship to the standard reduction algorithm?
  - "Topological simplification"... of *what?*
- Because of a beautiful structure
  - Standard algebraic techniques
  - Algorithm *falls out!*
  - Extends to all dimensions, all fields of coefficient

Persistence Complex

- Chain complex C_*

$$\ldots \rightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \rightarrow \ldots$$

- Filtration

$$K^0 \xrightarrow{f^0} K^1 \xrightarrow{f^1} K^2 \xrightarrow{f^2} \ldots$$

- Chain complexes

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \ldots$$

$$\begin{array}{ccccccc}
\partial_3 \downarrow & & \partial_3 \downarrow & & \partial_3 \downarrow & & \\
C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \ldots \\
\partial_2 \downarrow & & \partial_2 \downarrow & & \partial_2 \downarrow & & \\
C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \ldots \\
\partial_1 \downarrow & & \partial_1 \downarrow & & \partial_1 \downarrow & & \\
C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \ldots
\end{array}$$

Artin-Rees Construction

- Filtration: $K^0 \xrightarrow{f^0} K^1 \xrightarrow{f^1} K^2 \xrightarrow{f^2} \cdots$
- Grading $(K^0, K^1, K^2, \ldots) = \bigoplus K^i$
- If R is a ring, we endow $R[t]$ with standard grading $(t^n) = t^n \cdot R[t]$, $n \geq 0$
- Form a graded module over $R[t]$ where the action of t is

$$t \cdot (s^0, s^1, s^2, \ldots) = (0, f^0(s^0), f^1(s^1), f^2(s^2), \ldots)$$

- t simply shifts elements of the module up in the gradation.

Structure

- F, a field, $F[t]$ is a PID
- Graded $F[t]$-modules

$$\left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} F[t]\right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} F[t]/(t^{n_j})\right)$$

where Σ is the shift operator.

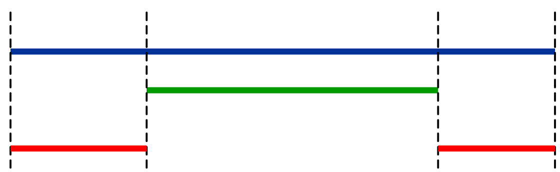
- Barcode: multiset of intervals
  - Half-infinite $\Sigma^{\alpha_i} F[t] \mapsto (\alpha^i, \infty)$
  - Finite $\Sigma^{\gamma_j} F[t]/(t^{n_j}) \mapsto (\gamma_j, \gamma_j + n_j)$

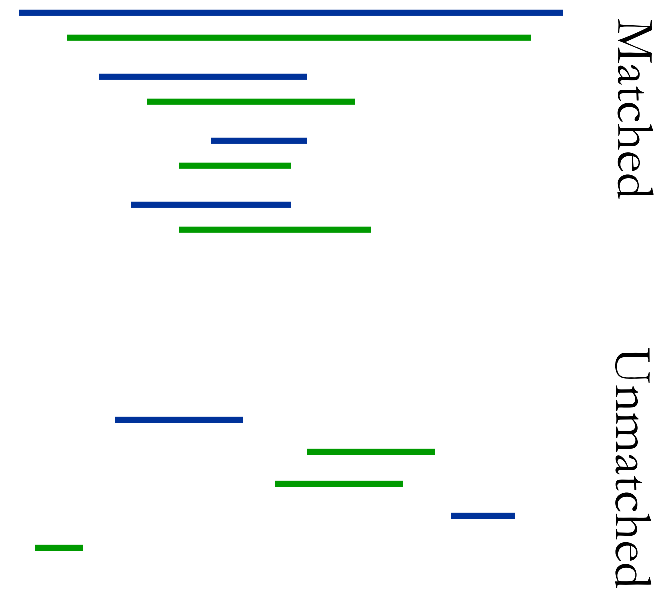
- No *nice* description if not a field

Barcode Metric

- Dissimilarity

$\delta(I, J) = |I \cup J - I \cap J|$

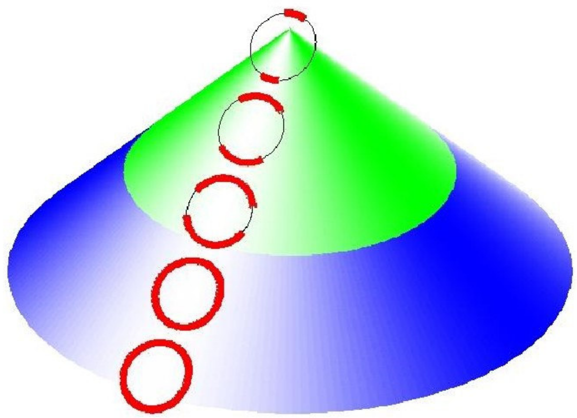




- Distance for matching M: $d_M(B_1, B_2)$
- Metric: $D(B_1, B_2) = \min_M d_M(B_1, B_2)$

Order doesn't matter…

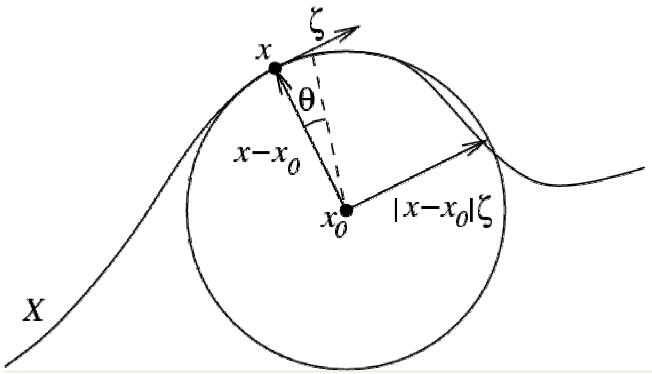
Filtered Tangent Complex



Second Order Contact

- We incrementally construct the tangent complex by looking at the curvature at each point.

- $(x, \zeta) \in T^0(X)$ has a circle of second order contact if $\exists\, x_0 \in \mathbb{R}^n$ such that $(x - x_0) \cdot v = 0$ for all v for which $(x, v) \in T^0(X)$ and



$$\lim_{\theta \to 0} \frac{d(x_0 + \cos\theta \cdot (x - x_0) + \sin\theta \cdot |x - x_0| \cdot \zeta, X)}{\theta^2} = 0.$$

Filtered Tangent Complex

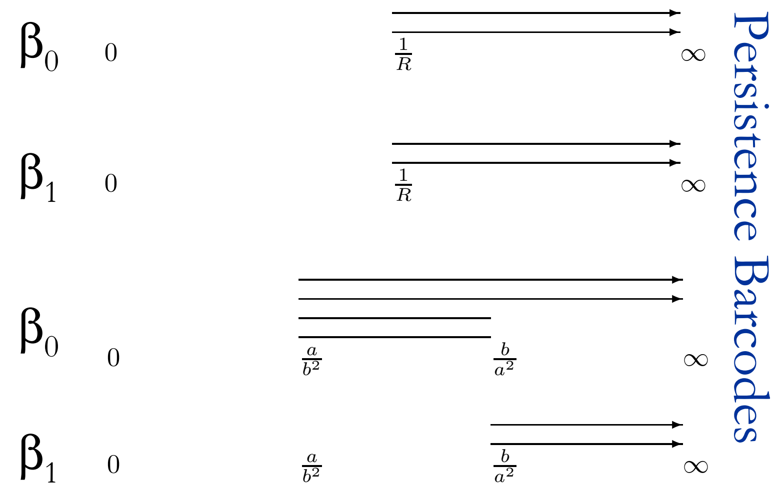
- Let $\rho(x, \zeta) = |x - x_0|$ be the radius the circle of second order contact, if it exists
- Let $T_\delta{}^0(X)$ be points $(x, \zeta) \in T^0(X)$ that have circle of second order contact with $1/\rho(x, \zeta) \le \delta$.
- Let $T_\delta(X)$ be the closure of $T_\delta{}^0(X)$.

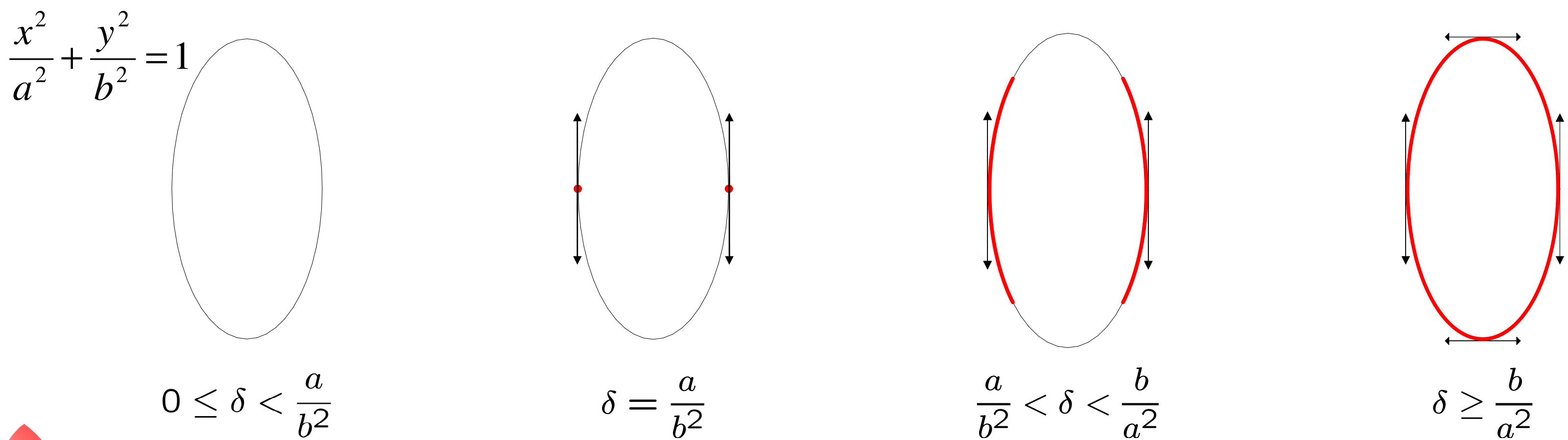
- The filtered tangent complex $T^{filt}(X)$ is the family $\{T_\delta(X)\}_{\delta \ge 0}$.
- The tame tangent complex $T^{tame}(X)$ is $\bigcup_\delta T_\delta(X)$.

Differentiating a Circle from an Ellipse

T^{filt}(circle of radius R) is simple: the entire complex (2 copies of circle) appears at $\delta = 1/R$.

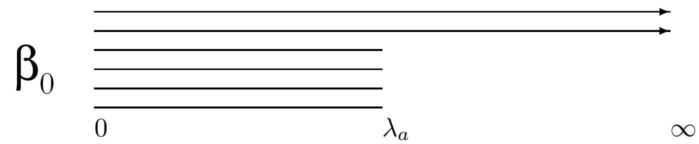
T^{filt}(ellipse) evolves through a stage with 4 components: points at *lower* curvature appear earlier.

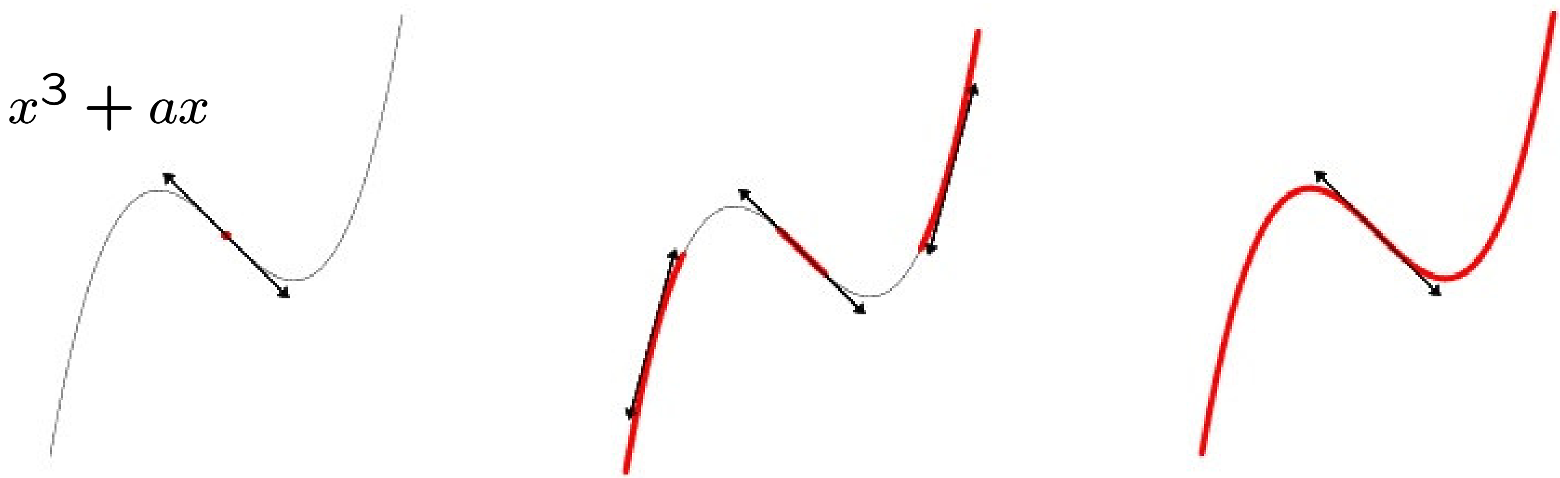




Parameterizing a Cubic

$$\lambda_a = \sqrt{\frac{-2a + \sqrt{5 + 9a^2}}{15}}$$

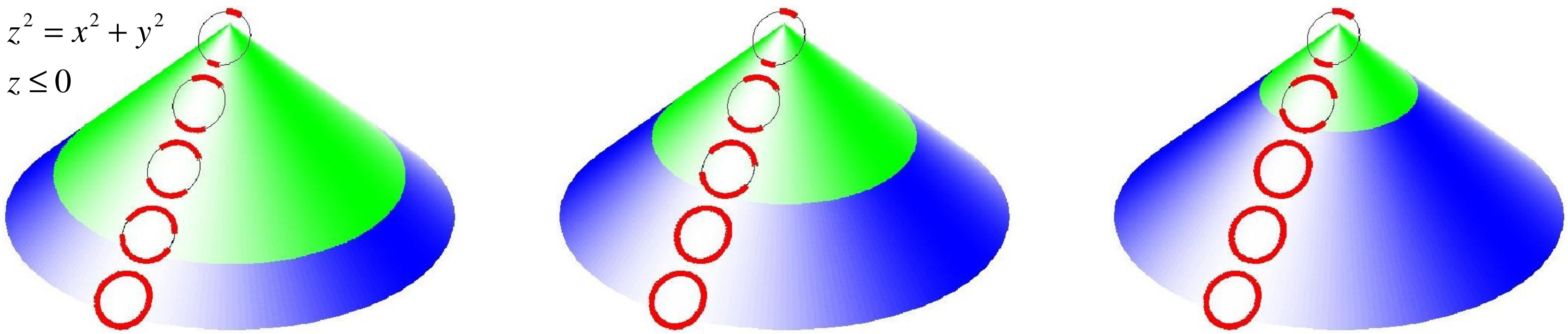




$\delta = 0$ $\quad 0 < \delta < \lambda_a$ $\quad \delta \geq \lambda_a$

Detecting a Cone Point

- Each regular point has a tangent direction with curvature zero. So at $\delta = 0$, the fiber at the cone point is two circles.
- The further away from cone point, the lower the max curvature, and the earlier the entire fiber is in $T_\delta(X)$.
- Each blue point has a circle as fiber, so the homology of $T_\delta(X)$ is always $\mathbb{S}^1 \times \mathbb{S}^1$.

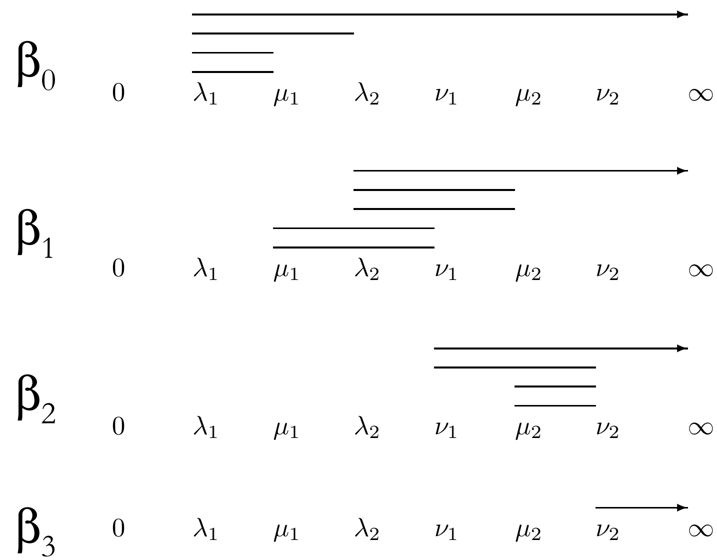


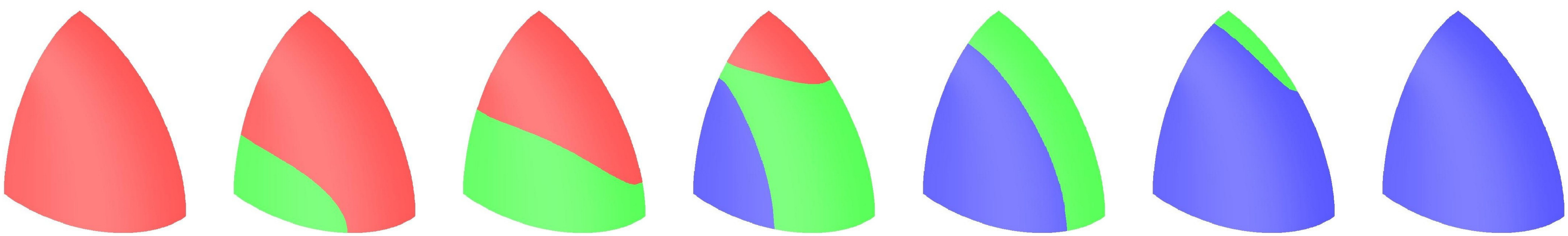
Blue points have a $\mathbb{S}^1$ as their fiber in $T_\delta(X)$, with δ increasing to the right

Describing an Ellipsoid

- T^{filt} depends on the ordering of principal curvatures κ_1 and κ_2 at the ellipsoid extrema
- We show the case $a^2c < b^3 < ac^2$ for ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$





(empty, two arcs, circle)

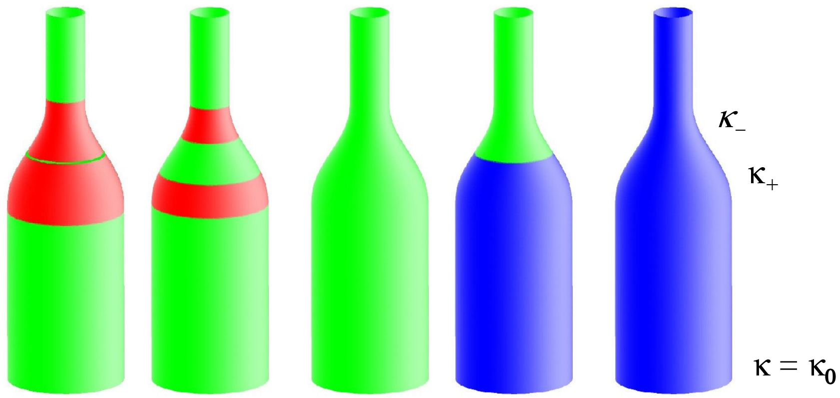
Shape Recognition

Distinguishing Geometric Surfaces

- $T^{tame}(X)$
- Descriptor: $(\beta_1,\beta_2,\beta_3)$
- Independent of coordinate system, deformations
- Hard features

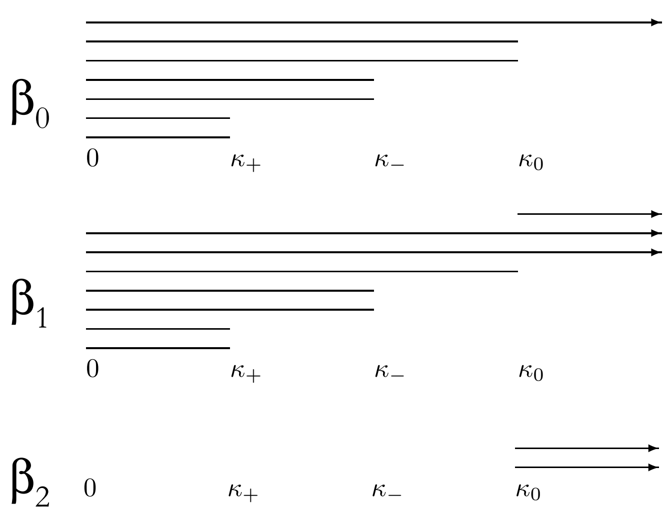
Surface	β_1	β_2	β_3
Sphere	1	1	1
Cone	2	2	0
Cube	24	0	0
Tetrahedron	13	0	0
Pyramid	21	0	0
Cylinder	4	3	0

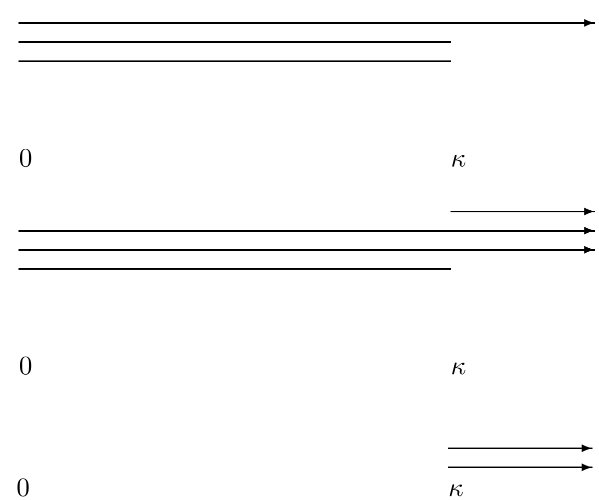
Differentiating a Bottle from a Glass



(empty, two arcs, circle)

- Glass = bottom of Bottle
- T^{filt} depends on neck curvatures κ_+, κ_- and cross-sectional curvature $\kappa = \kappa_0$ at the bottom.





Computing Distances

- Given two bottles of type analyzed with parameters $(\kappa_+, \kappa_-, \kappa_0)$ and $(\kappa'_+, \kappa'_-, \kappa'_0)$, the β_0 distance is $2(|\kappa_+ - \kappa'_+| + (|\kappa_- - \kappa'_-| + (|\kappa_0 - \kappa'_0|)$

- Given two glasses with curvatures κ and κ', the distance is $2|\kappa - \kappa'|$.

- Given a bottle and a glass, there are three cases:

1.	$0 < 2\kappa \leq \kappa_+ + \kappa_-$	$2|\kappa_+ - \kappa| + 2\kappa_- + 2\kappa_0$
2.	$\kappa_+ + \kappa_- \leq 2\kappa \leq \kappa_- + \kappa_0$	$2\kappa_+ + 2|\kappa_- - \kappa| + 2\kappa_0$
3.	$2\kappa \geq \kappa_- + \kappa_0$	$2\kappa_+ + 2\kappa_- + 2|\kappa_0 - \kappa|$

Conclusion

- Persistent homology + geometry-rich derived space = barcode
- Barcode metric
  - Comparison
  - Matching
  - Classification
- Recent work: curve point cloud data (Anne's talk)
- Future work:
  - Surface PCD
  - Complexes with lower intrinsic dimension
  - Learning Algorithms