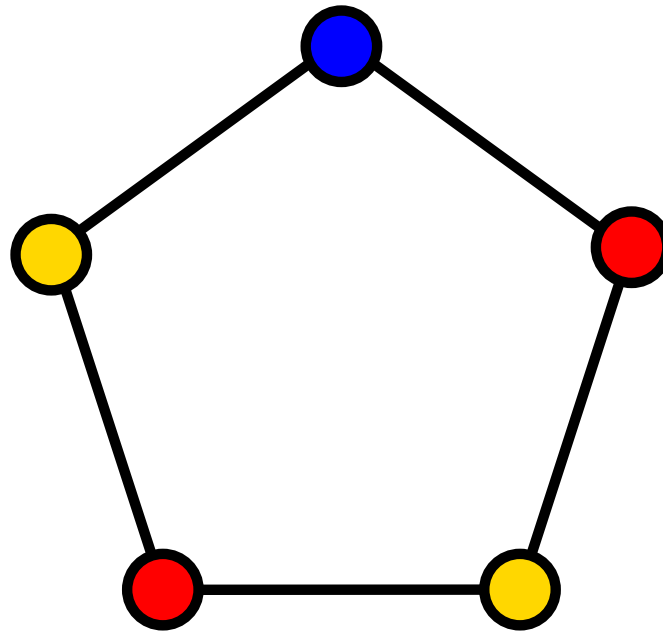


Graph Coloring Manifolds

Frank H. Lutz

(TU Berlin/ZIB)



1. Graph coloring
2. The neighborhood complex and Lovász' proof of the Kneser conjecture
3. Hom complexes
4. Sphere bundles
5. Graph coloring manifolds
6. Flag 1-spheres
7. Flag 2-spheres

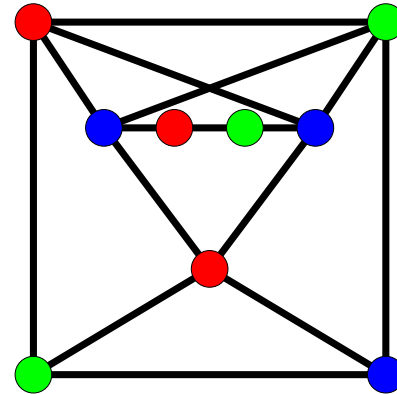
(Joint work with Péter Csorba.)

1. Graph coloring

graph coloring and lower bounds

Let $G = (V, E)$ be a graph with vertex set V and edge set E .

▷ $\chi(G) :=$ **chromatic number** of G

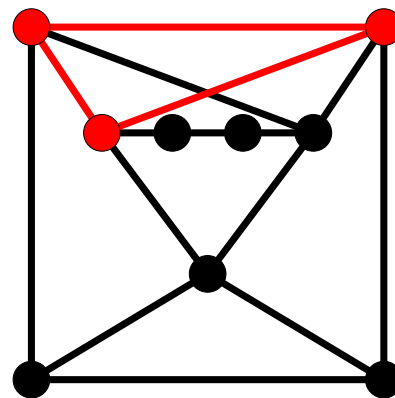


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Let $G = (V, E)$ be a graph with vertex set V and edge set E .

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- ▷ $\omega(G) :=$ **clique number** of G

$$\omega(G) \leq \chi(G)$$



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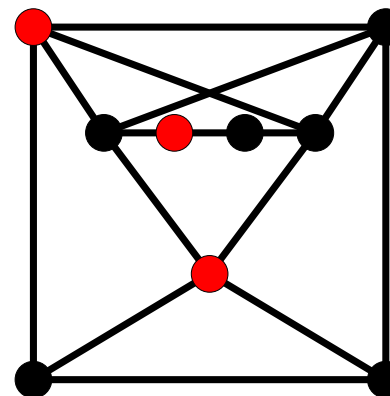
▷ $\chi(G) :=$ **chromatic number** of G

▷ $\omega(G) :=$ **clique number** of G

$$\omega(G) \leq \chi(G)$$

▷ $\alpha(G) :=$ **stable set number** of G

$$\frac{|V|}{\alpha(G)} \leq \chi(G)$$



THEOREM

KARP (1972)

It is NP-complete to determine $\chi(G)$.

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COOK (1971)

It is NP-complete to determine $\alpha(G)$
(and also $\omega(G) = \alpha(\overline{G})$).

Difficulties with lower bounds:

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- ▷ often rather weak
- ▷ hard to compute
- ▷ yield good results only in specific cases

Topological approach:

$G \mapsto K(G)$ simplicial complex/cell complex

“Compute” topological invariants of $K(G)$,
e.g., connectivity.

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- ▷ $K(G) = \mathcal{N}(G)$, neighborhood complex (simplicial), Lovász
- ▷ $K(G) = B(G)$, box complex (simplicial), Matoušek and Ziegler
- ▷ $K(G) = \text{Hom}(H, G)$, Hom complex (cellular), Lovász

2. The neighborhood complex and Lovász' proof of the Kneser conjecture

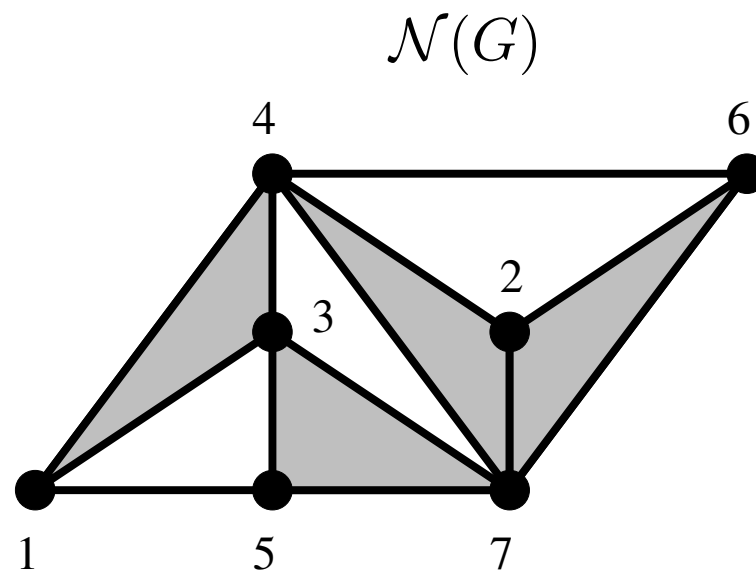
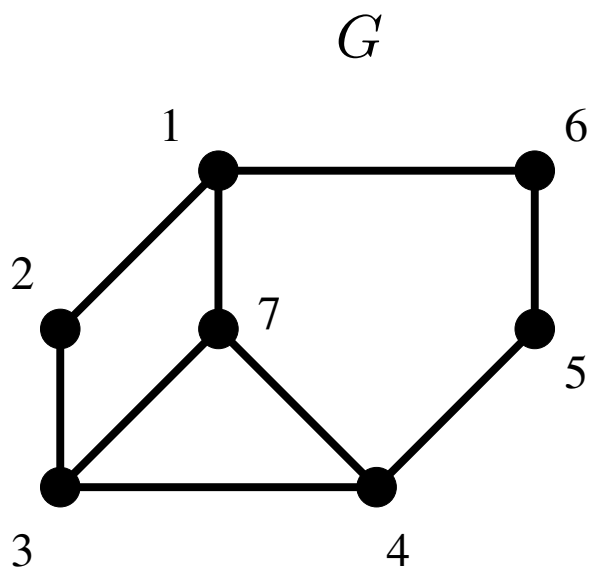
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▷ simplicial complex:

vertices := nodes of G

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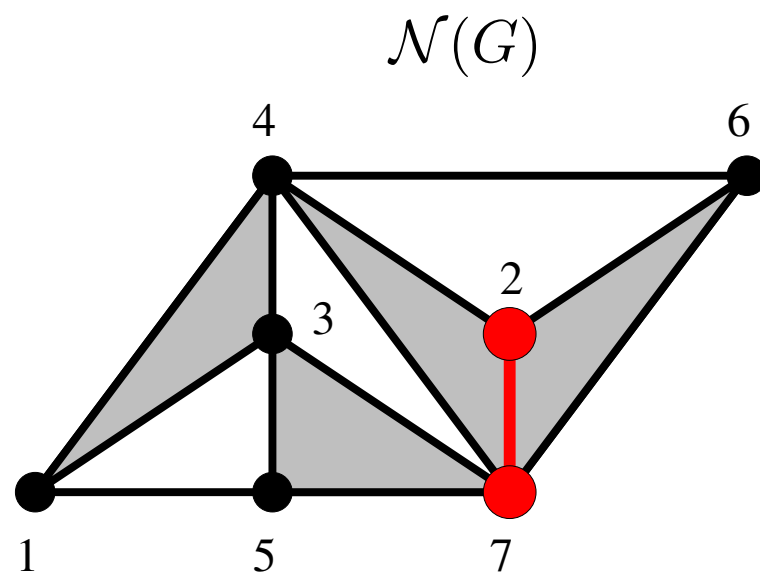
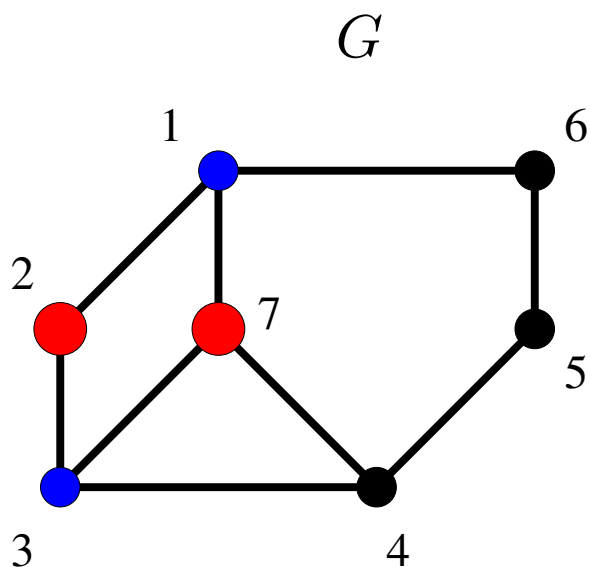


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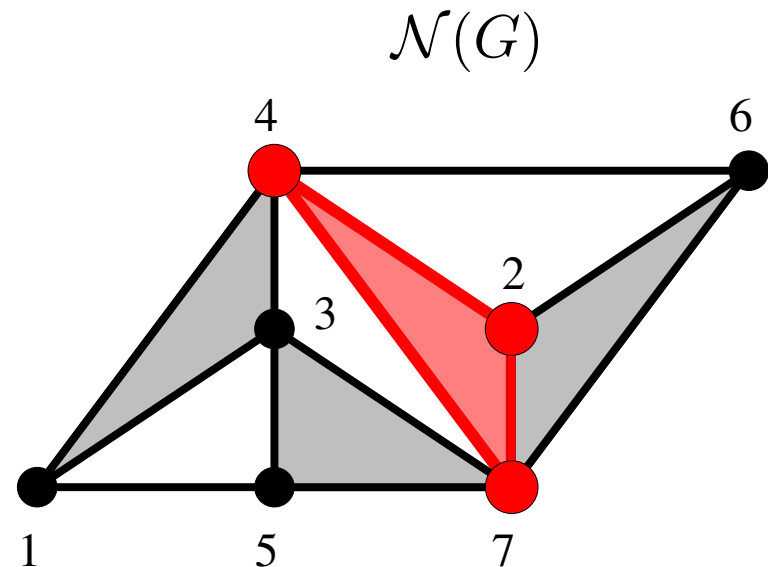
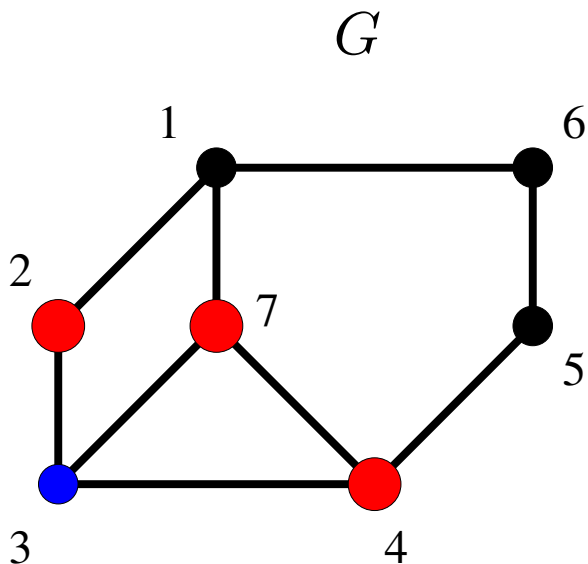


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Kneser's conjecture (1955)

If we partition the k -element subsets of an n -element set into $n - 2k + 1$ classes, then one of the classes contains two disjoint k -element subsets.

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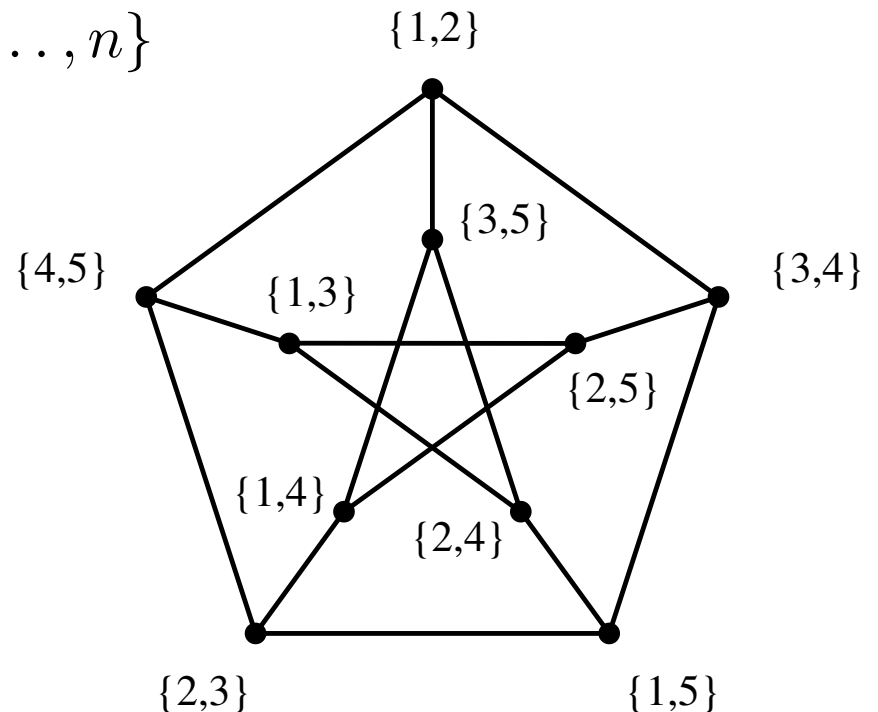
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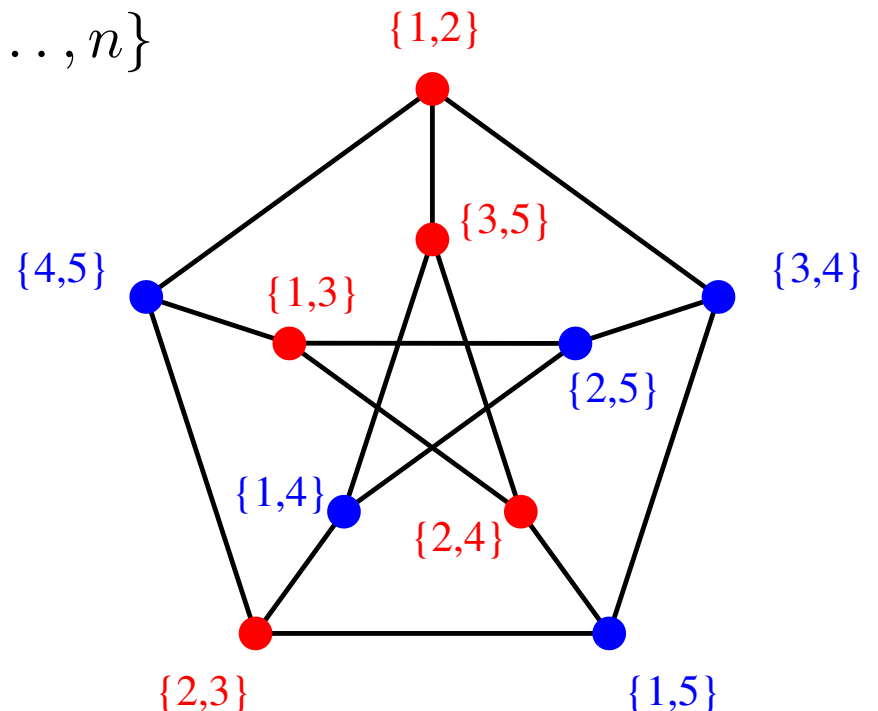
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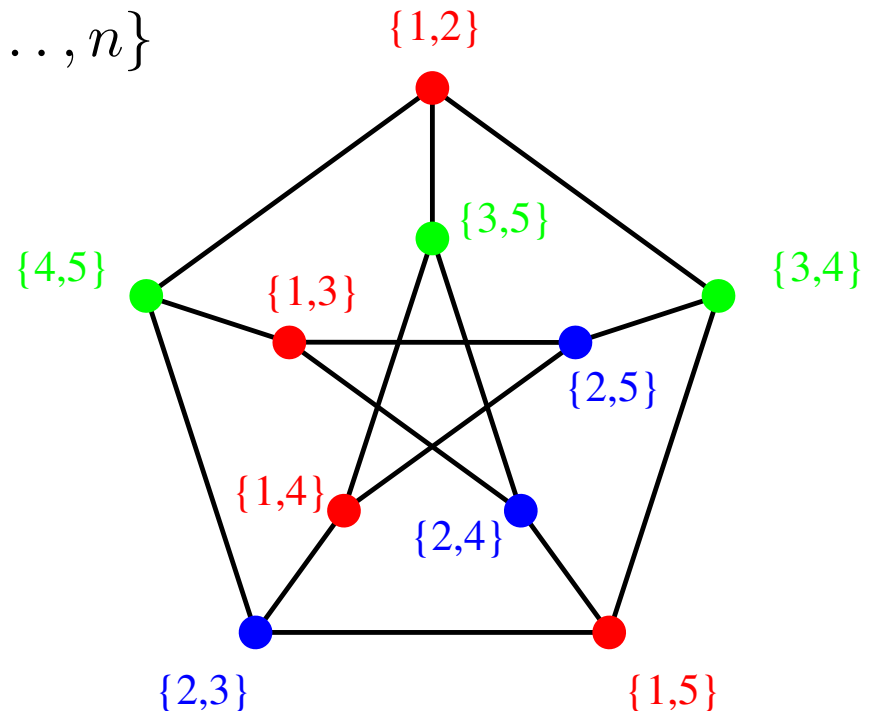
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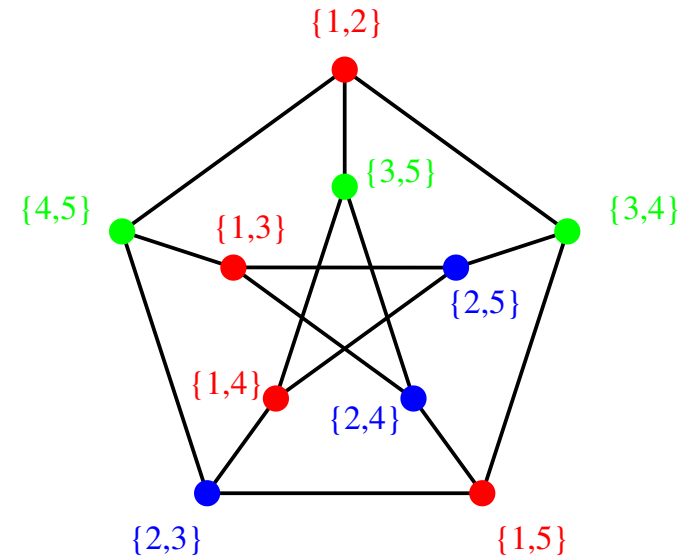


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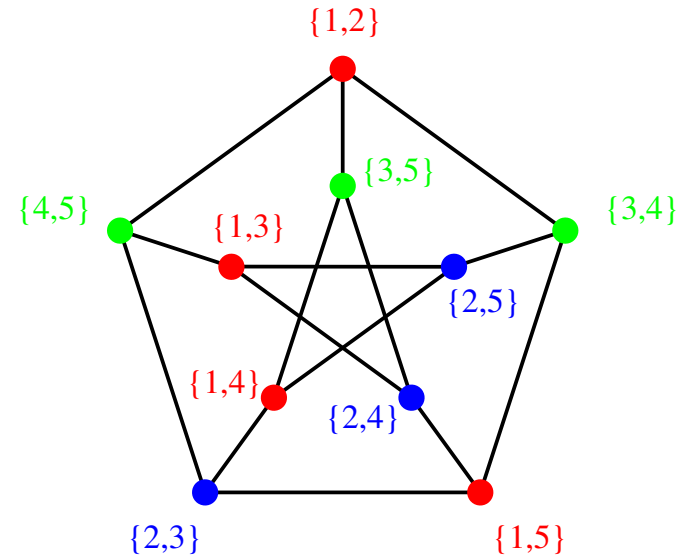
$K_i :=$ collection of k -sets with first element i .

Any two sets from the $n - 2k + 2$ classes

$$K_1, K_2, \dots, K_{n-2k+1}, K_{n-2k+2} \cup \dots \cup K_{n-k+1},$$

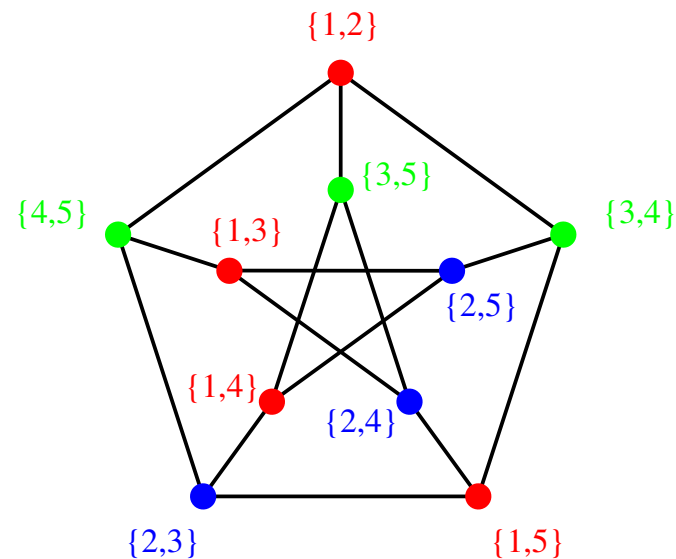
have a common element [since $n - (n - 2k + 1) = 2k - 1$].

► For $n = 3k - 1$: $\omega(KG_{3k-1,k}) = 2$.



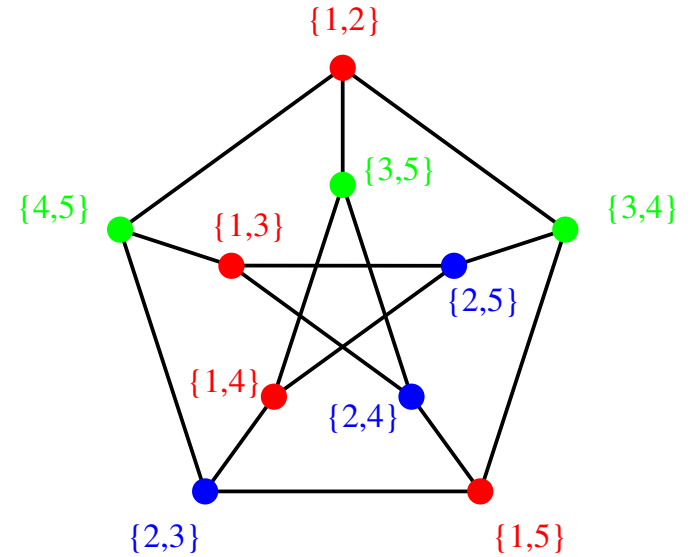
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Lovász' proof of the Kneser conjecture

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The chromatic number of the Kneser graph $KG_{n,k}$ is $\chi(KG_{n,k}) = n - 2k + 2$.

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▷ Lovász uses the theorem of Borsuk-Ulam.

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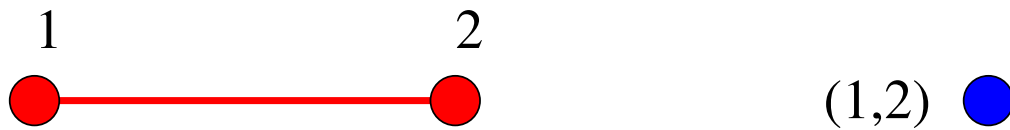
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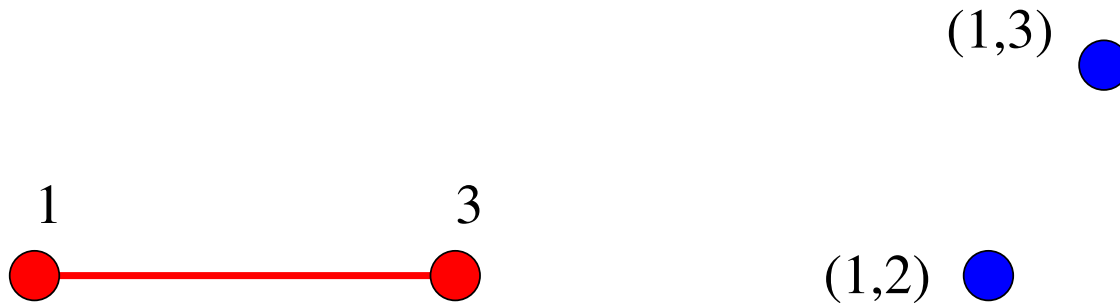
$\mathcal{N}(KG_{n,k})$ is $(n - 2k - 1)$ -connected.

3. Hom complexes

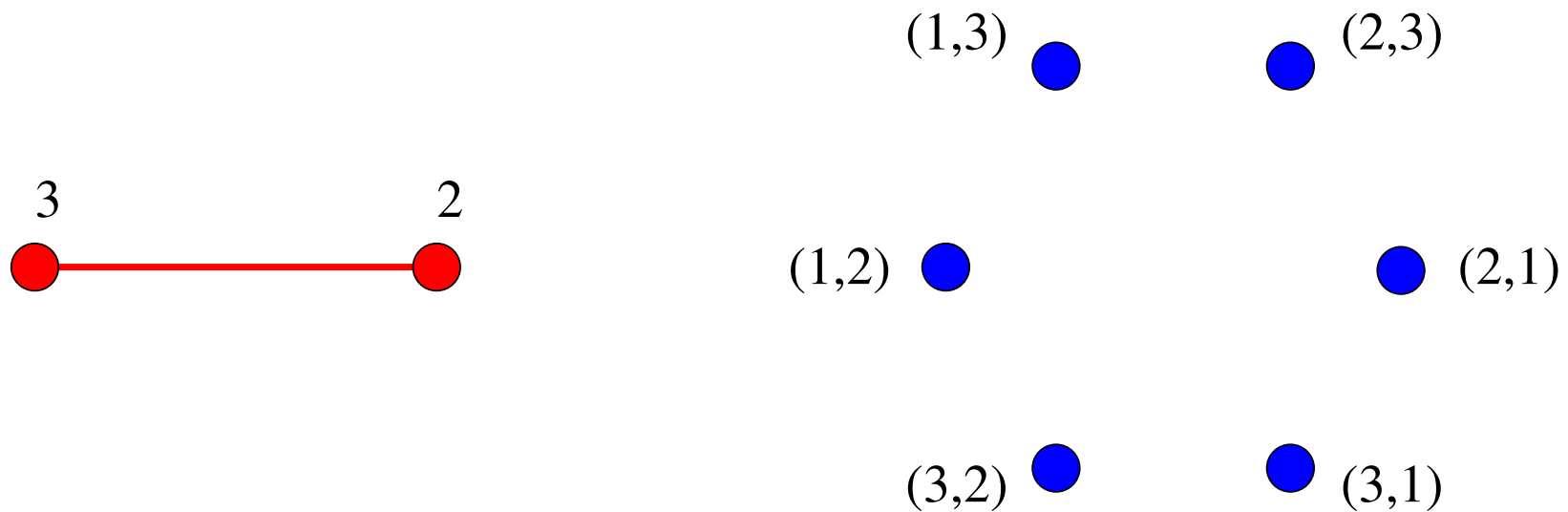
The Hom complex $\text{Hom}(K_2, K_3)$:



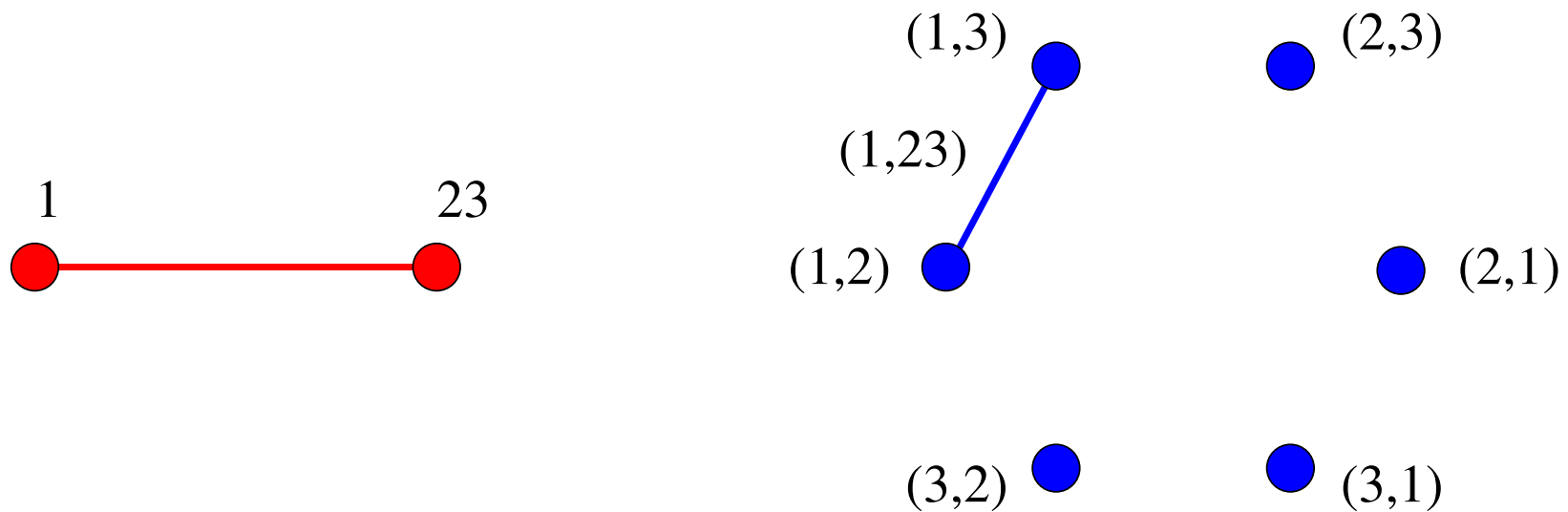
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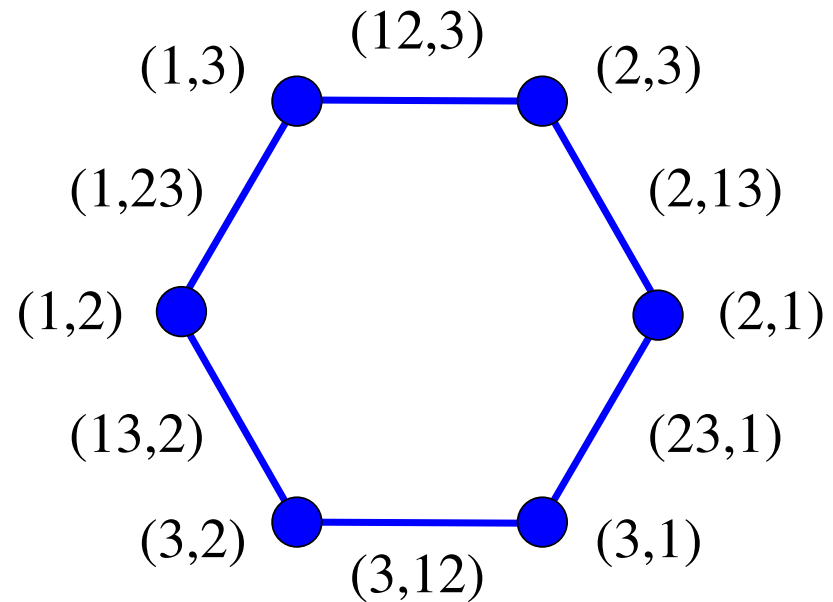
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The *Hom complex* $\text{Hom}(H, G)$ of two graphs H and G :

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▷ polyhedral complex with:

vertices := graph homomorphisms $\phi : V(H) \rightarrow V(G)$,

i.e., for $e = (i, j) \in E(H)$

we have $(\phi \times \phi)(e) = (\phi(i), \phi(j)) \in E(G)$

cells := indexed by all functions $\eta : V(H) \rightarrow 2^{V(G)} \setminus \{\emptyset\}$,

i.e., for $(i, j) \in E(H)$

we have for all $\tilde{i} \in \eta(i)$ and $\tilde{j} \in \eta(j)$

that $(\tilde{i}, \tilde{j}) \in E(G)$

some properties of Hom complexes

LOVÁSZ/BABSON & KOZLOV (2003):

- ▷ $\text{Hom}(K_2, G)$ is homotopy equivalent to $\mathcal{N}(G)$.

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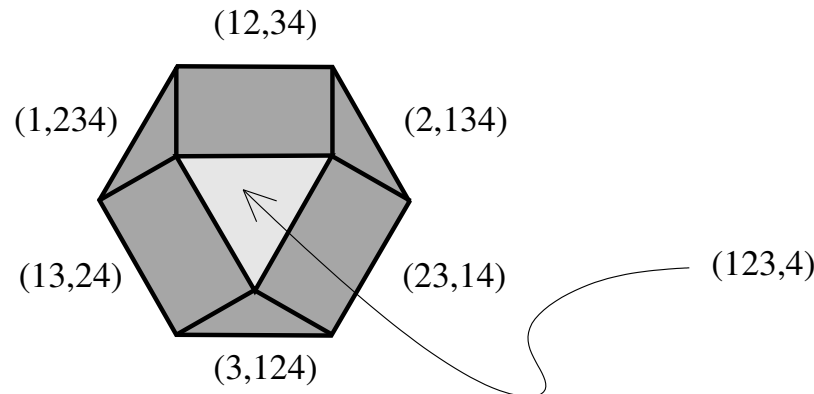
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- ▷ $\text{Hom}(G \dot{\cup} H, K) = \text{Hom}(G, K) \times \text{Hom}(H, K)$.

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- ▷ $\text{Hom}(G \dot{\cup} H, K) = \text{Hom}(G, K) \times \text{Hom}(H, K)$.
- ▷ $\text{Hom}(K_2, K_{n+2}) \cong S^n$,

$\text{Hom}(K_2, K_3) \subseteq \text{Hom}(K_2, K_4)$:



Lovász' conjecture & Csorba's conjecture

THEOREM

BABSON & KOZLOV (2003)

If $\text{Hom}(C_{2r+1}, G)$ is k -connected, then $\chi(G) \geq k + 4$.

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If $\text{Hom}(C_{2r+1}, G)$ is k -connected, then $\chi(G) \geq k + 4$.

$H_*(\text{Hom}(C_5, K_n)) = (\mathbb{Z}, 0, \dots, 0, \mathbb{Z}, \mathbb{Z}, *, \dots, *)$ if n is odd,
 $H_*(\text{Hom}(C_5, K_n)) = (\mathbb{Z}, 0, \dots, 0, \mathbb{Z}_2, 0, *, \dots, *)$ if n is even.

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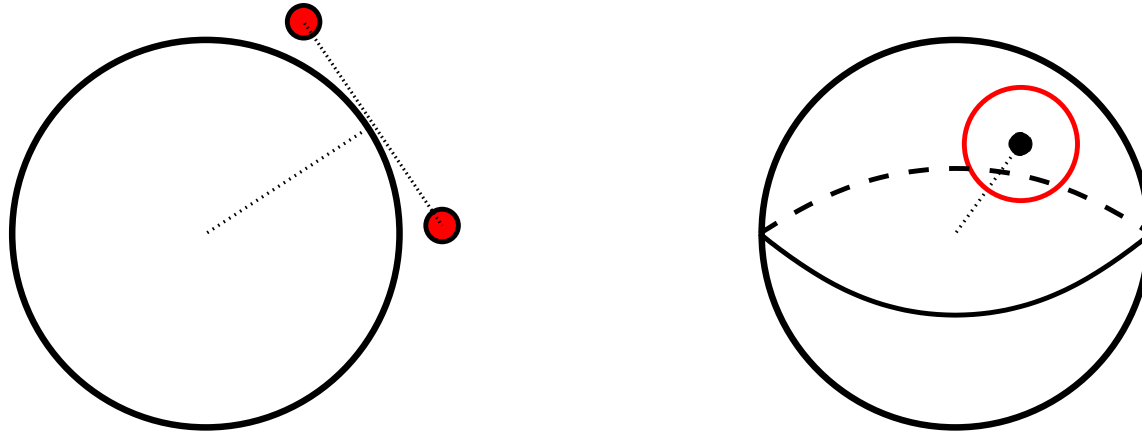
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CONJECTURE

CSORBA (2003)

The Hom complex $\text{Hom}(C_5, K_{n+2})$ is homeomorphic to the Stiefel manifold $V_{n+1,2}$.



$$\begin{aligned} V_{n+1,2} &= \{ (v_1, v_2) \mid v_1, v_2 \in \mathbb{R}^{n+1}, \|v_1\| = \|v_2\| = 1, \langle v_1, v_2 \rangle = 0 \} \\ &= \text{unit tangent bundle to } S^n. \end{aligned}$$

$$V_{2,2} = S^0 \times S^1.$$

$$V_{3,2} = \mathbb{RP}^3.$$

Hom(C_5, K_4), a 3D puzzle

Date: Sun, 22 Jun 2003 10:41:38 +0200 (MET DST)

From: Peter Csorba <pcsorba@inf.ethz.ch>

Dear Frank,

I have a question about the running time of your BISTELLAR Software.
I ran it on the enclosed complex (2160 vertex, 3 dimensional)
for few days (400 Mhz)...

best, Peter

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[1, 3, 295, 375], [1, 11, 40, 332], [1, 11, 40, 1861],
[1, 11, 179, 383], [1, 11, 179, 1922], [1, 11, 303, 332],
[1, 11, 303, 383], [1, 11, 1842, 1861], [1, 11, 1842, 1922],
[1, 34, 36, 84], [1, 34, 36, 328], [1, 34, 40, 332],
....

12960 tetrahedra!

▷ compute *homology* with

`polymake` (Gawrilow, Joswig et al.),

`Simplicial Homology` (Dumas, Heckenbach, Saunders, Welker):

$$H_*(\text{Hom}(C_5, K_4)) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}).$$

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▷ check *vertex links*:

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- ▷ check *vertex links*:

$\text{Hom}(C_5, K_4)$ is a manifold.

- ▷ compute *fundamental group* with `GAP` (Groups, Algorithms, and Programming):

$$\pi_1(\text{Hom}(C_5, K_4)) = \mathbb{Z}_2.$$

Thurston's geometrization conjecture implies:

$\mathbb{RP}^3$ is the only 3-manifold with fundamental group $\mathbb{Z}_2$.

[Proof(?) by Perelman, 2003.]

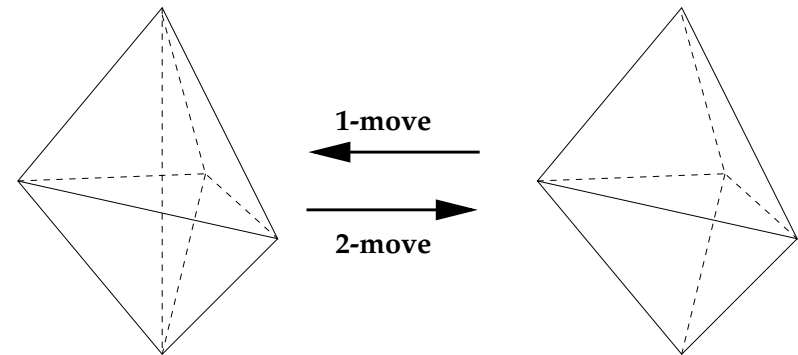
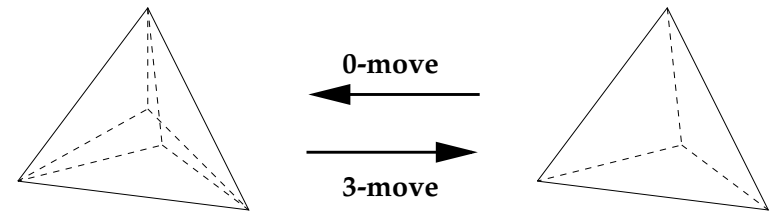
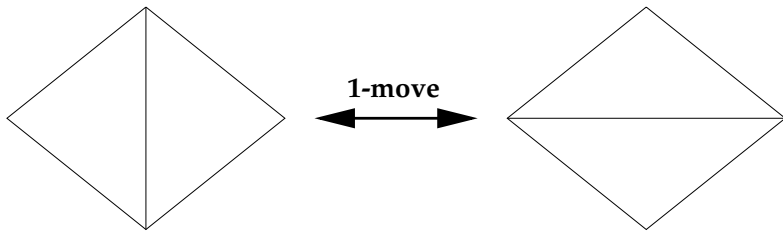
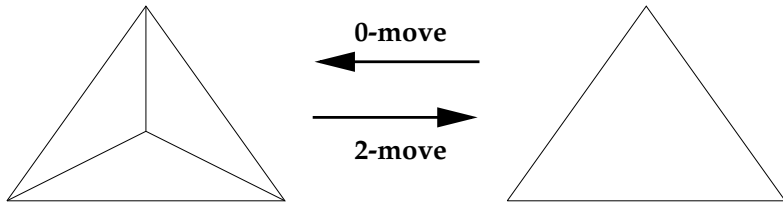
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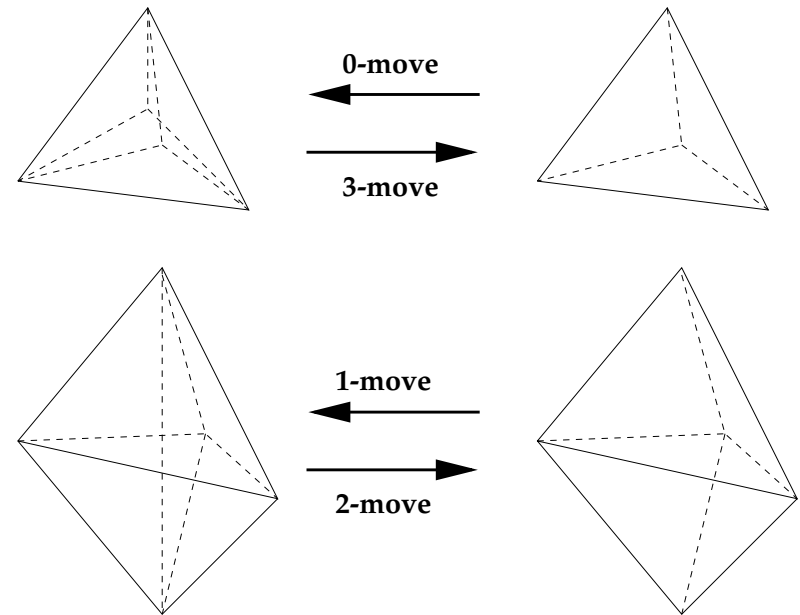
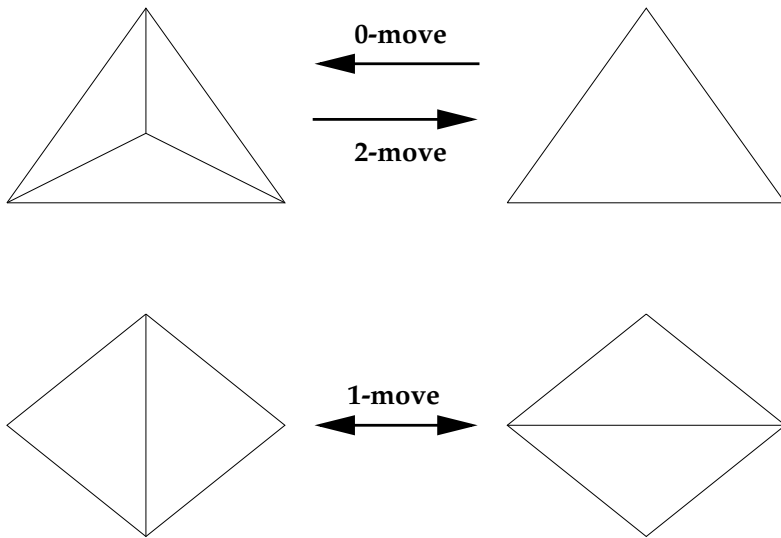
[Proof(?) by Perelman, 2003.]

Try alternative proof for $\text{Hom}(C_5, K_4) = \mathbb{RP}^3$:
recognition with bistellar flips.

bistellar flips

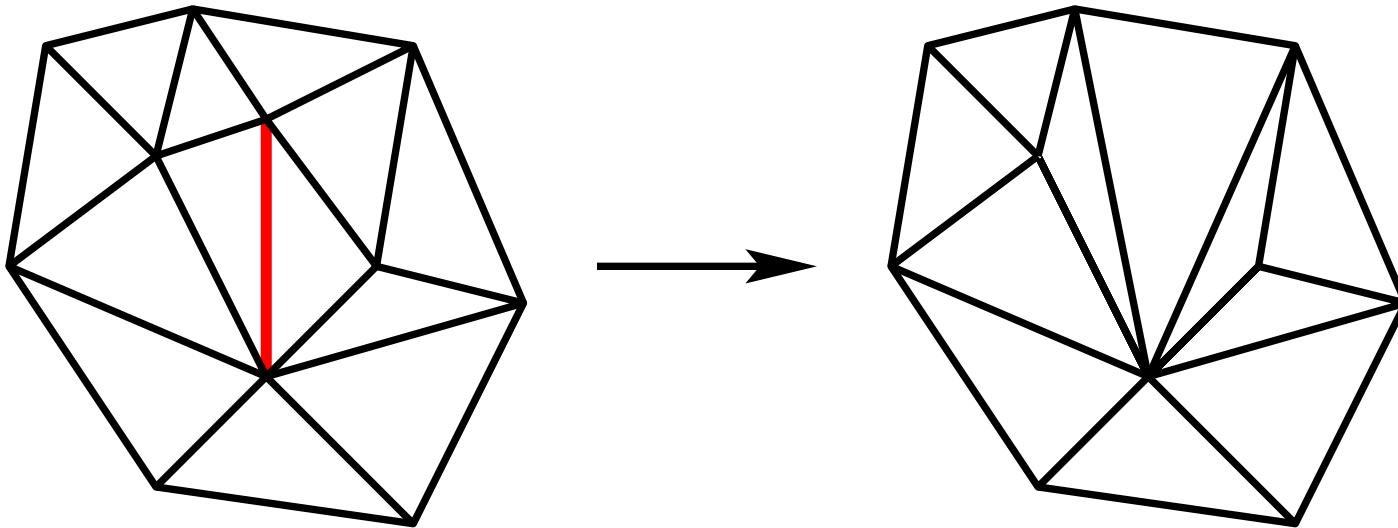


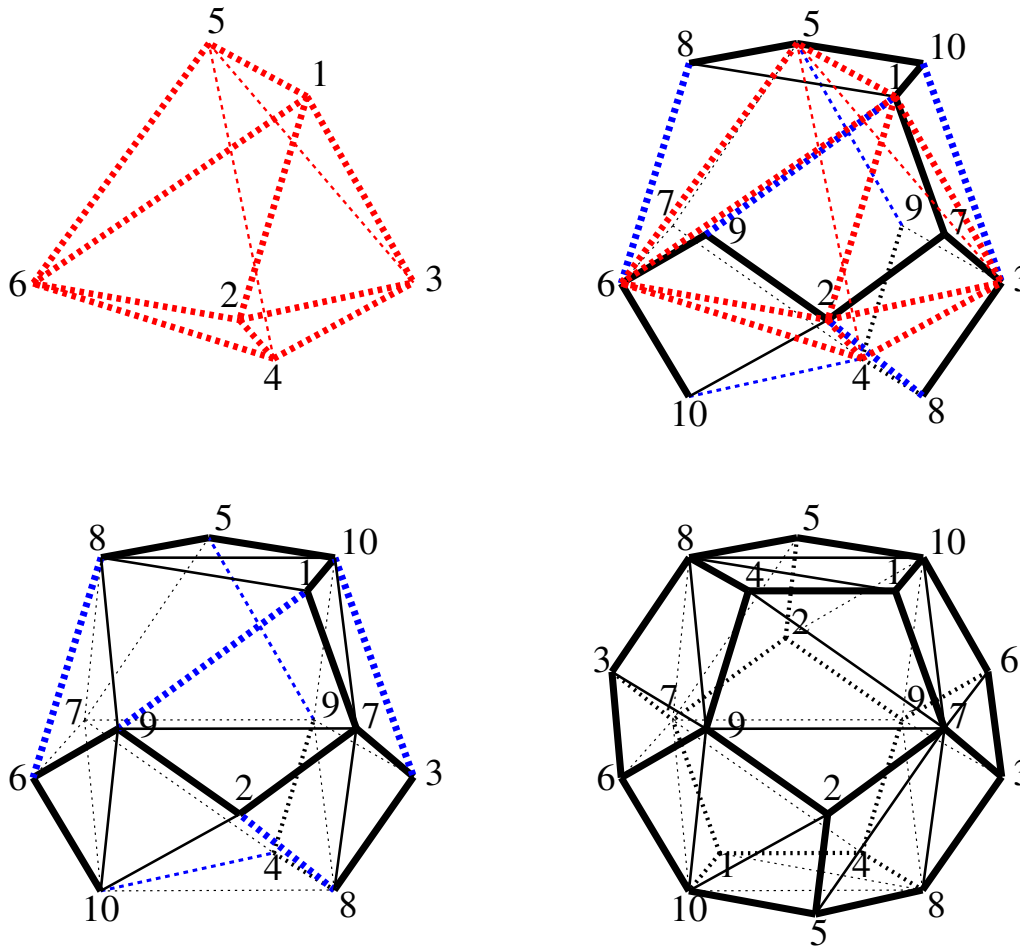
bistellar flips



Recognition works well for triangulations of S^3 and $\mathbb{RP}^3$ with up to 200 vertices.

For triangulations of S^3 and $\mathbb{RP}^3$
with up to 3000 vertices: need edge contractions.





minimal 11-vertex triangulation of $\mathbb{RP}^3$ [Walkup, 1970]

4. Sphere bundles

THEOREM

CSORBA & L. (2003)

$\text{Hom}(C_5, K_{n+2})$ is an S^{n-1} -bundle over S^n .

THEOREM

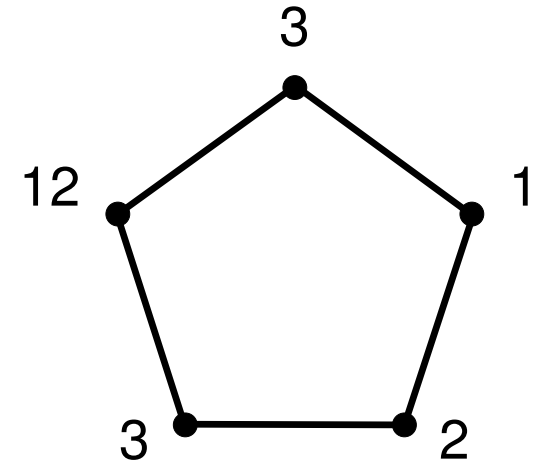
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$\text{Hom}(C_5, K_{n+2})$ is an S^{n-1} -bundle over S^n .

Proof:

- ▷ $\text{Hom}(C_5, K_{n+2})$ is a pure cell complex of dimension $d = 2(n + 2) - 5$.

Example: $(12, 3, 1, 2, 3)$ is a facet of $\text{Hom}(C_5, K_3)$.



THEOREM

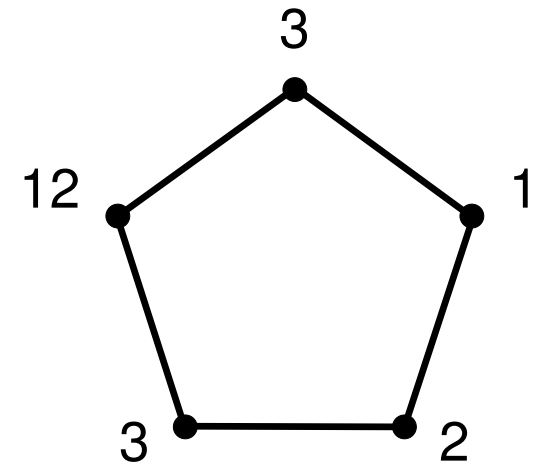
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$\text{Hom}(C_5, K_{n+2})$ is an S^{n-1} -bundle over S^n .

Proof:

- ▶ $\text{Hom}(C_5, K_{n+2})$ is a pure cell complex of dimension $d = 2(n + 2) - 5$.

Example: $(12,3,1,2,3)$ is a facet of $\text{Hom}(C_5, K_3)$.



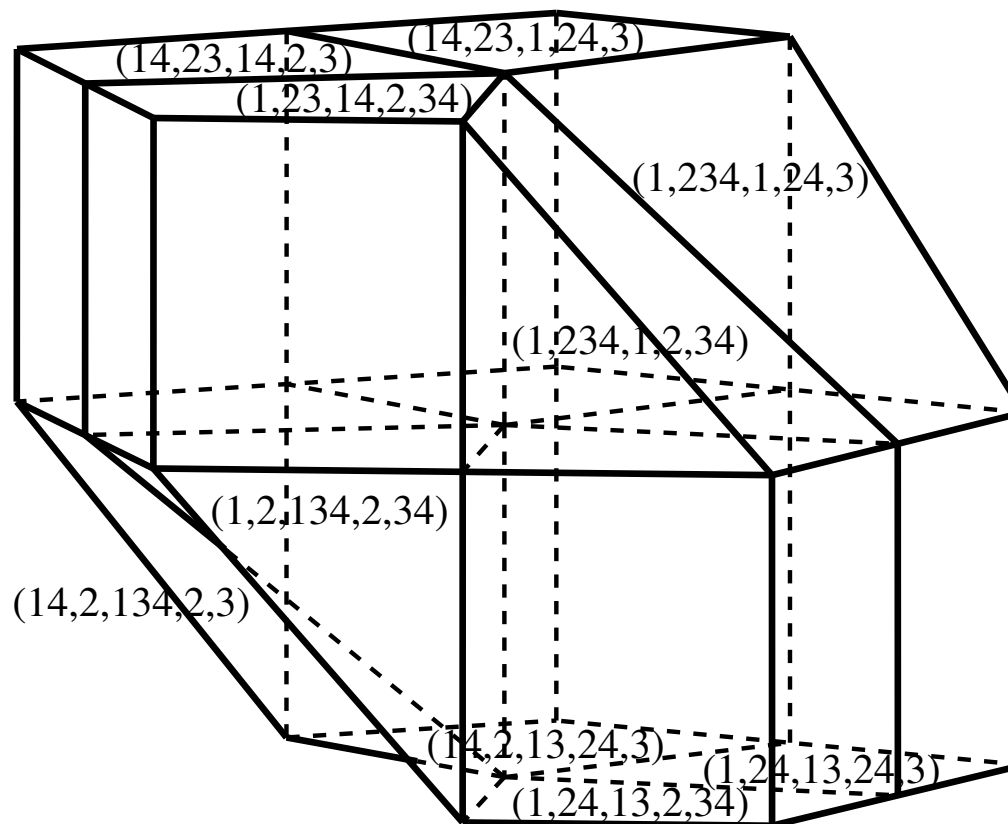
- ▶ $\text{Hom}(C_5, K_{n+2})$ is a pseudomanifold, i.e., every $(d - 1)$ -face is contained in exactly two facets.

Example: $(\cancel{12}, 3, 1, 2, 3)$.

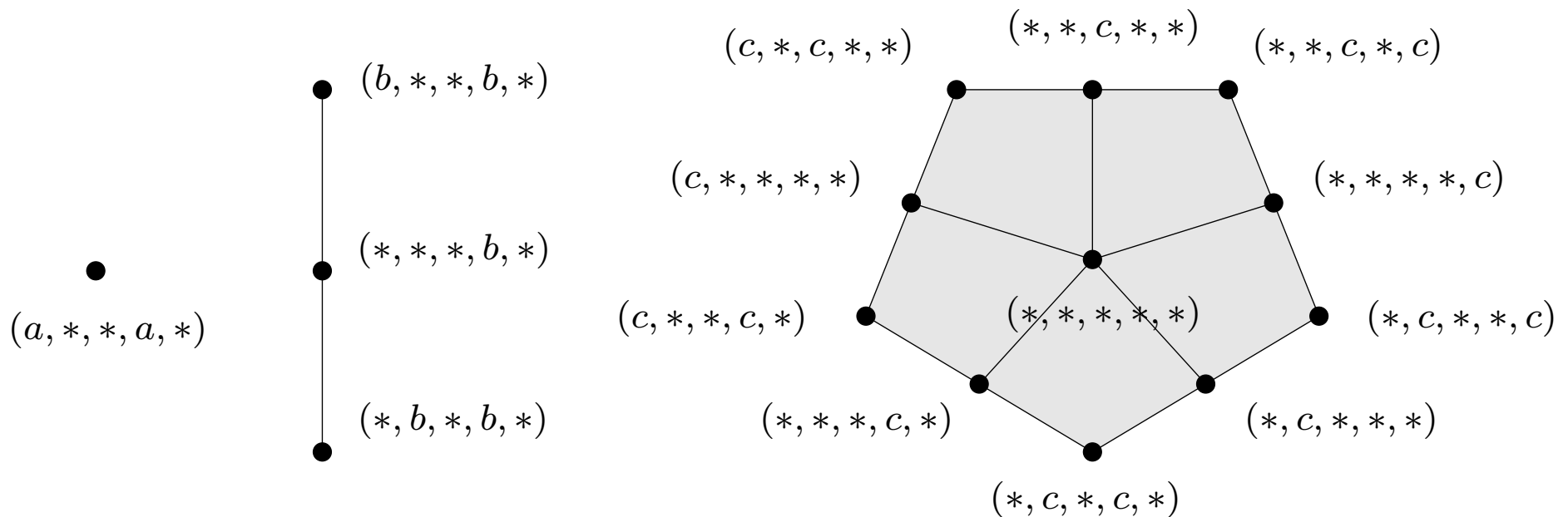
- ▷ $\text{Hom}(C_5, K_{n+2})$ is a PL-manifold,
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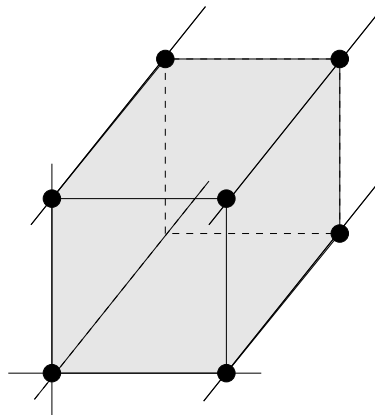
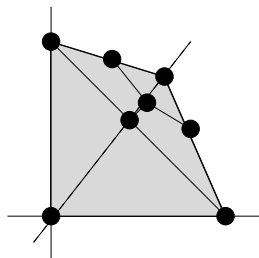
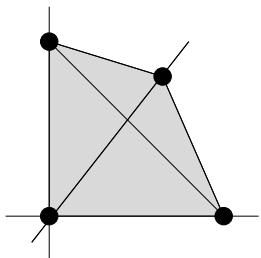
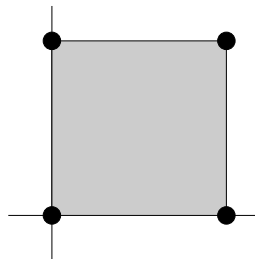
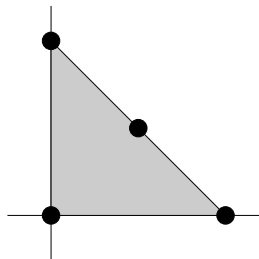
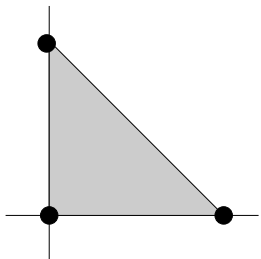
Example: Vertex-star of the vertex $(1, 2, 1, 2, 3)$ in $\text{Hom}(C_5, K_4)$:



Cubical complex $X := X_1 \times \cdots \times X_n$:

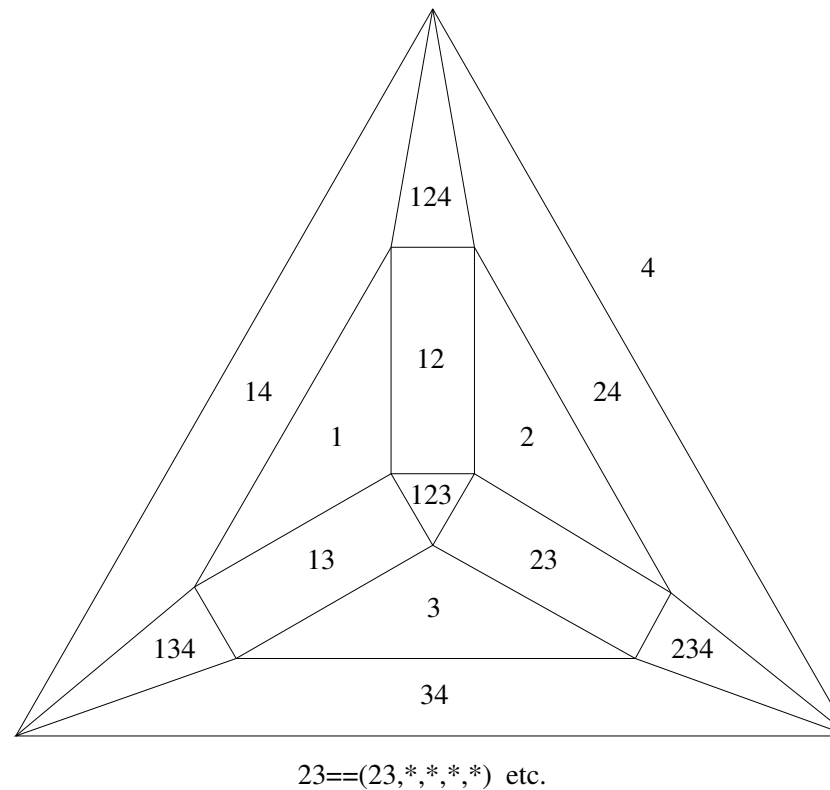


Subdivision of the vertex-stars in $\text{Hom}(C_5, K_{n+2})$:



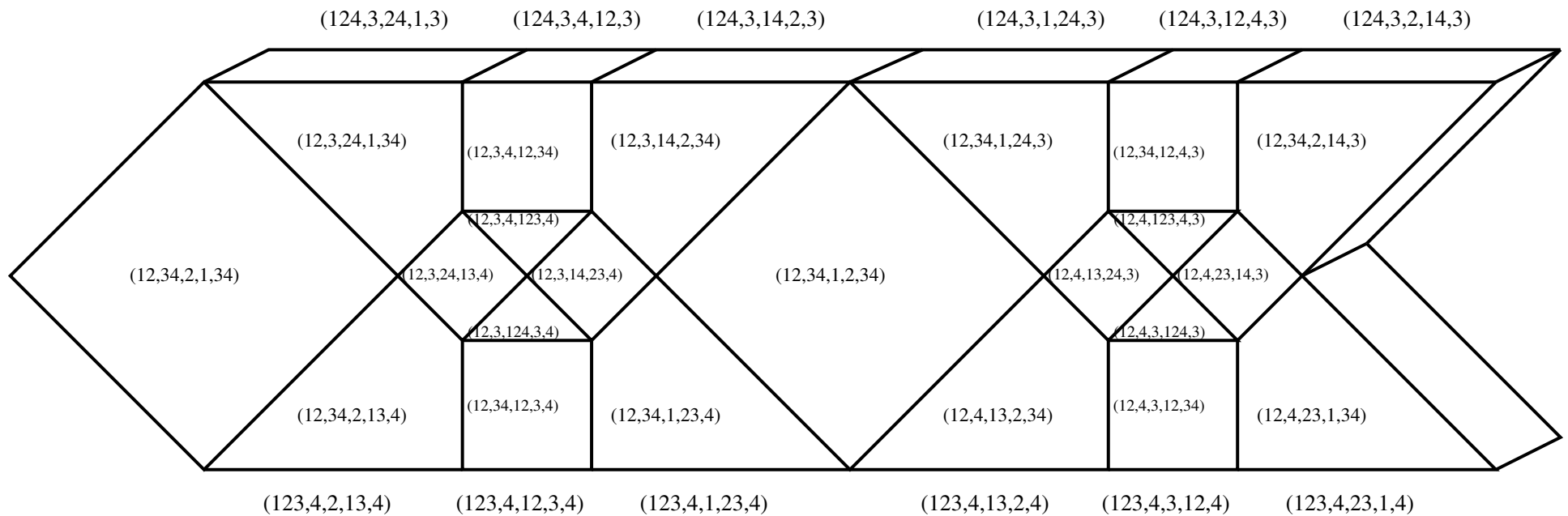
▷ $\text{Hom}(C_5, K_{n+2})$ is an S^{n-1} -bundle over S^n .

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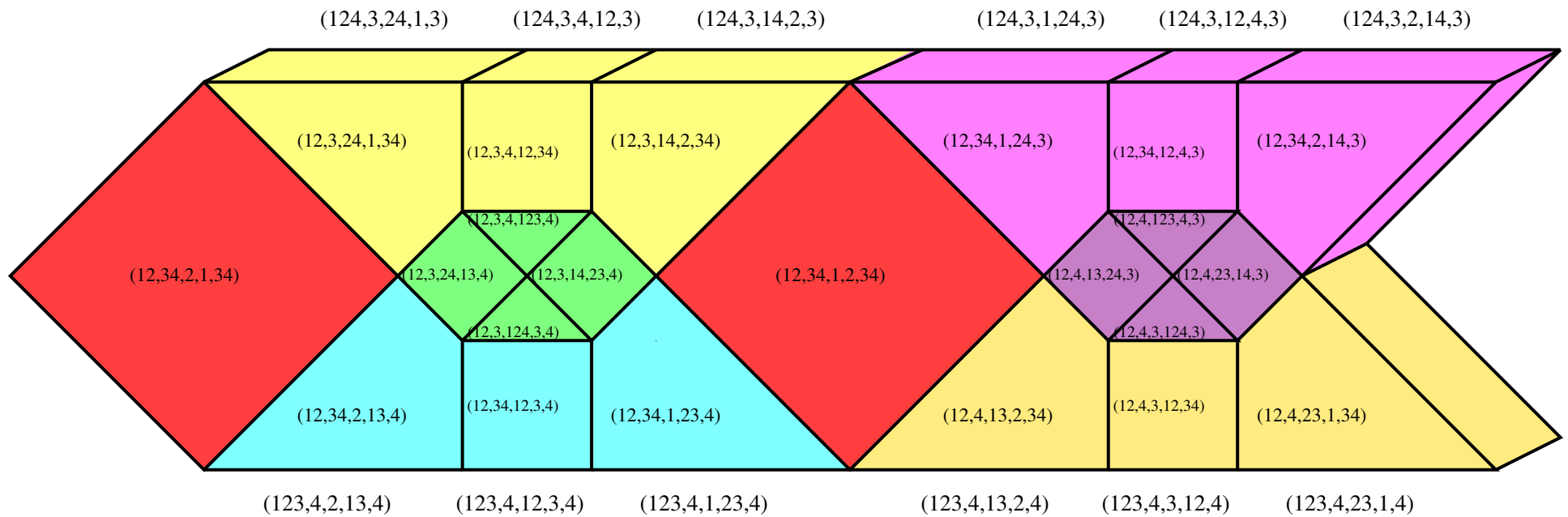


Cell-decomposition of the base sphere S^2 in $\text{Hom}(C_5, K_4)$.

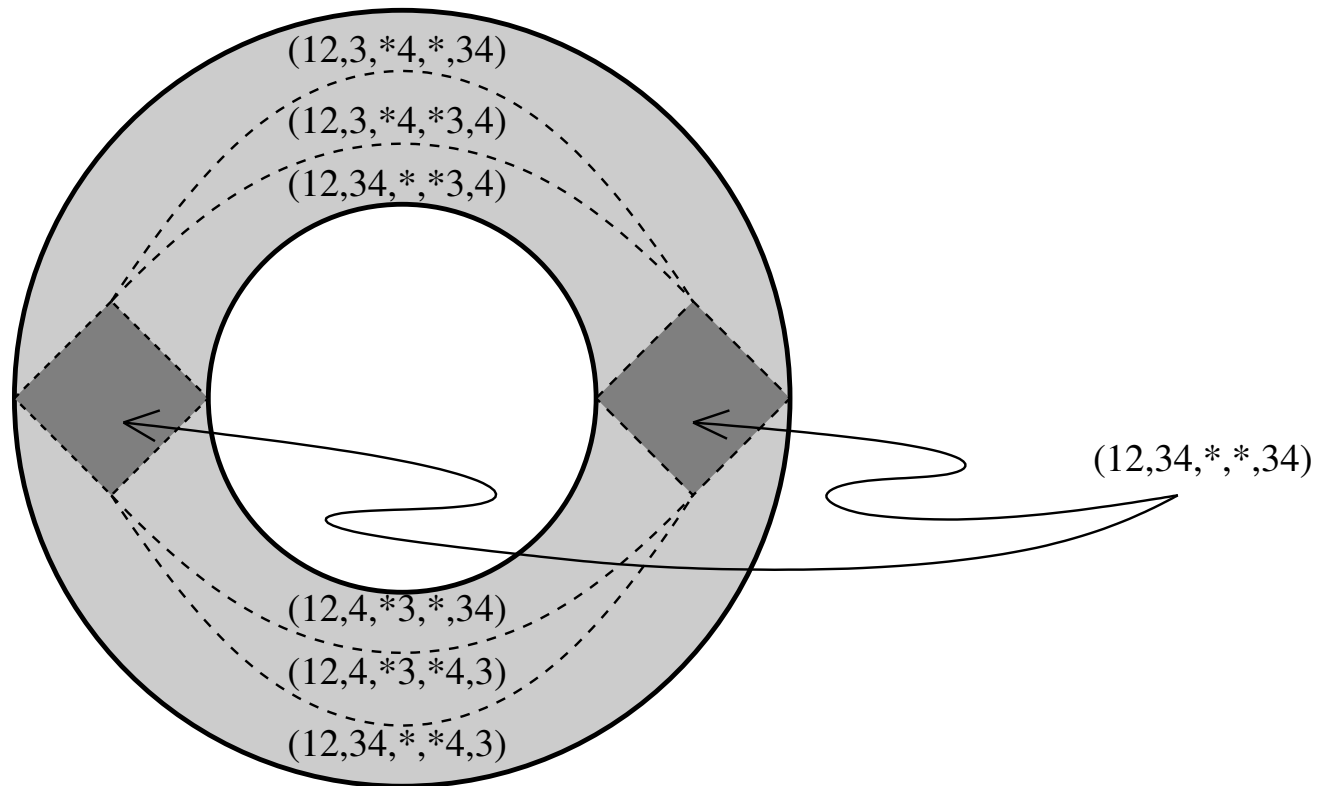
Collection of facets of the form $(12, *, *, *, *)$ in $\text{Hom}(C_5, K_4)$:



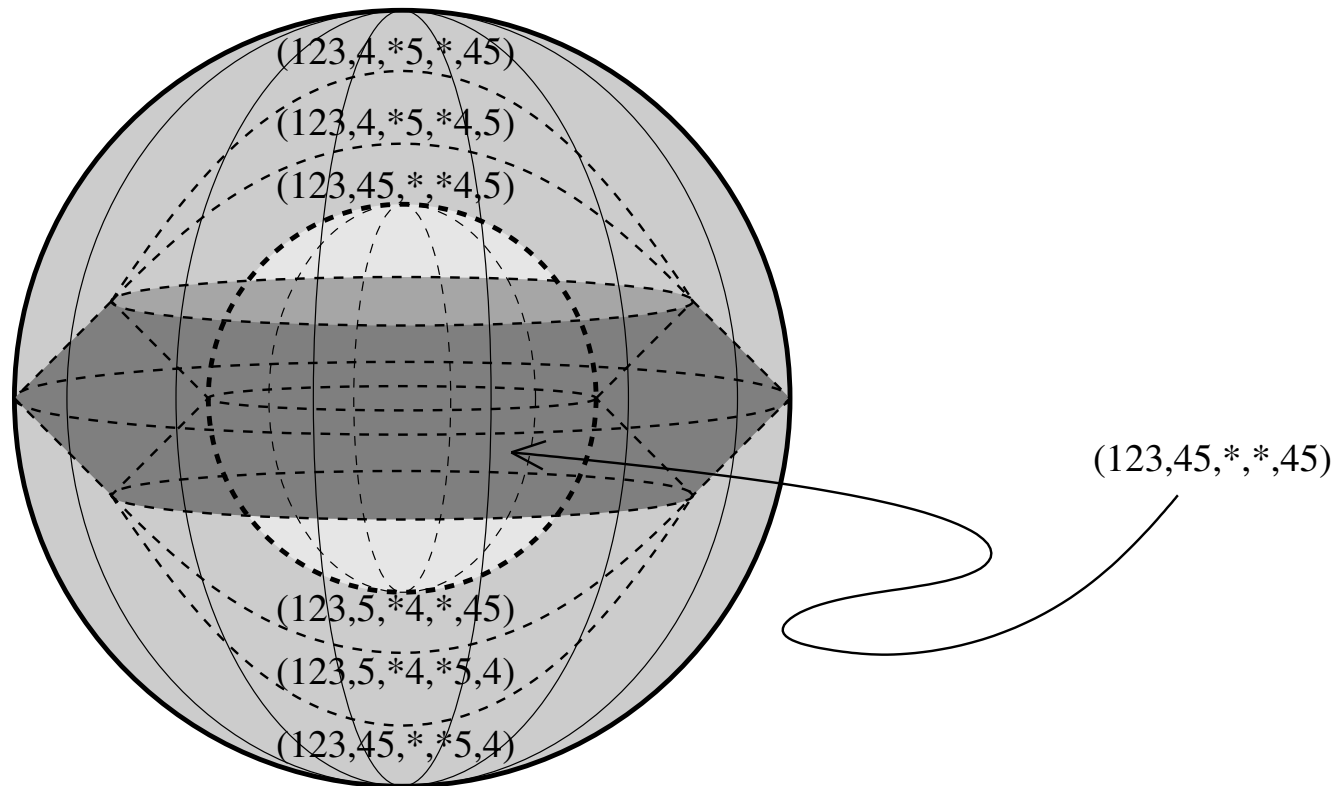
Collection of facets of the form $(12, *, *, *, *)$ in $\text{Hom}(C_5, K_4)$:



Decomposition of $(12, *, *, *, *) \subseteq \text{Hom}(C_5, K_4)$ into sub-collections of cells:



Decomposition of $(123, *, *, *, *) \subseteq \text{Hom}(C_5, K_5)$ into sub-collections of cells:



Two approaches to show that $\text{Hom}(C_5, K_{n+2}) \cong V_{n+1,2}$:

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- ▷ Find coordinates for vertices on $V_{n+1,2}$.

Two approaches to show that $\text{Hom}(C_5, K_{n+2}) \cong V_{n+1,2}$:

- ▷ Find coordinates for vertices on $V_{n+1,2}$.
- ▷ Use topological classification theorems:

$$\text{Hom}(C_5, K_3) = V_{2,2} = S^0 \times S^1,$$

$$\text{Hom}(C_5, K_4) = V_{3,2} = \mathbb{RP}^3,$$

$$\text{Hom}(C_5, K_5) = V_{4,2} = S^2 \times S^3 \quad [\pi_2(SO(3)) = 0],$$

$$\text{Hom}(C_5, K_6) = V_{5,2} \quad [\text{Escher \& Crowley (2003)}],$$

$$\text{Hom}(C_5, K_9) = V_{8,2} = S^6 \times S^7 \quad [\pi_6(SO(7)) = 0].$$

5. Graph coloring manifolds

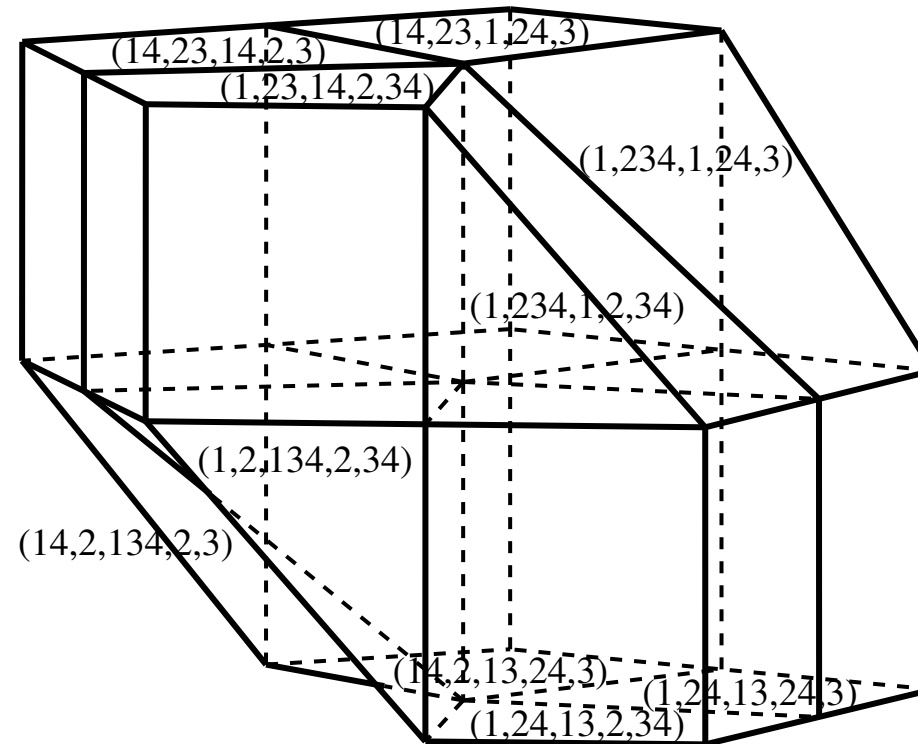
When is $\text{Hom}(H, K_n)$ a manifold?

For which graphs is $\text{Hom}(H, K_n)$ a manifold?

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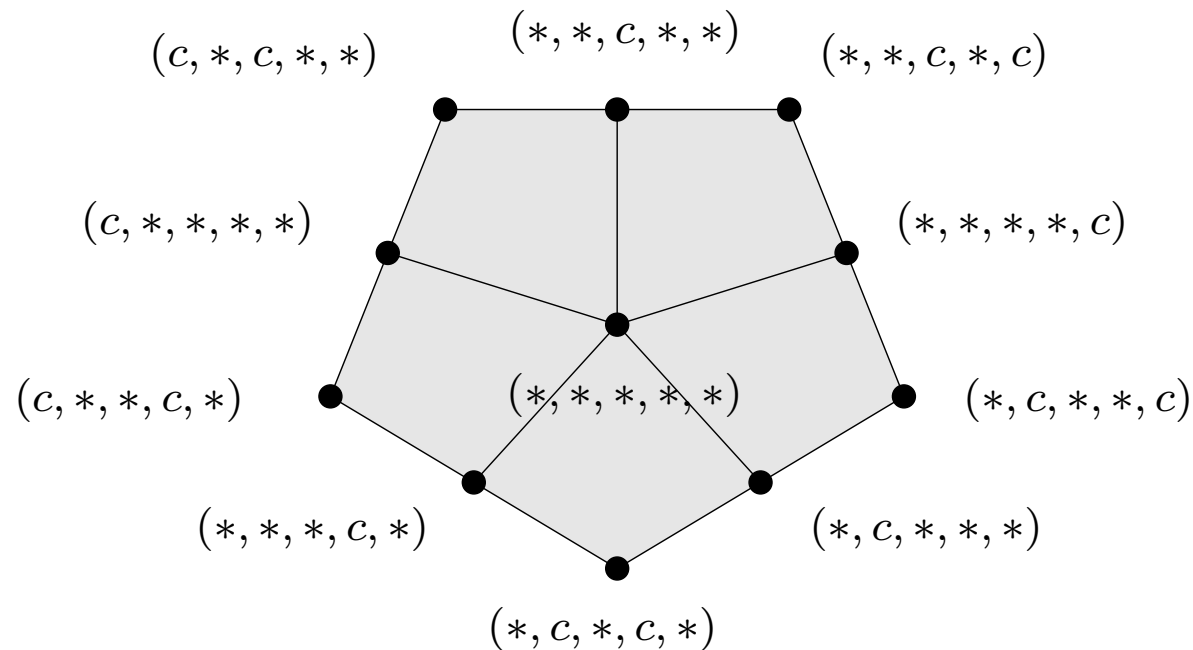
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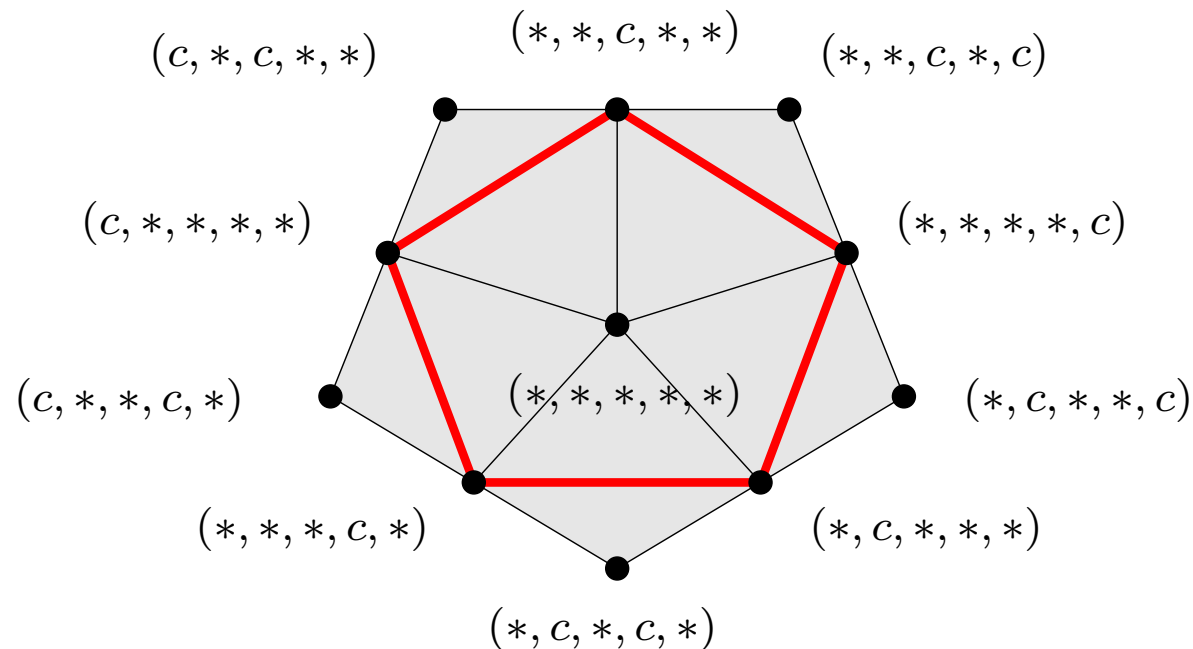
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We need:

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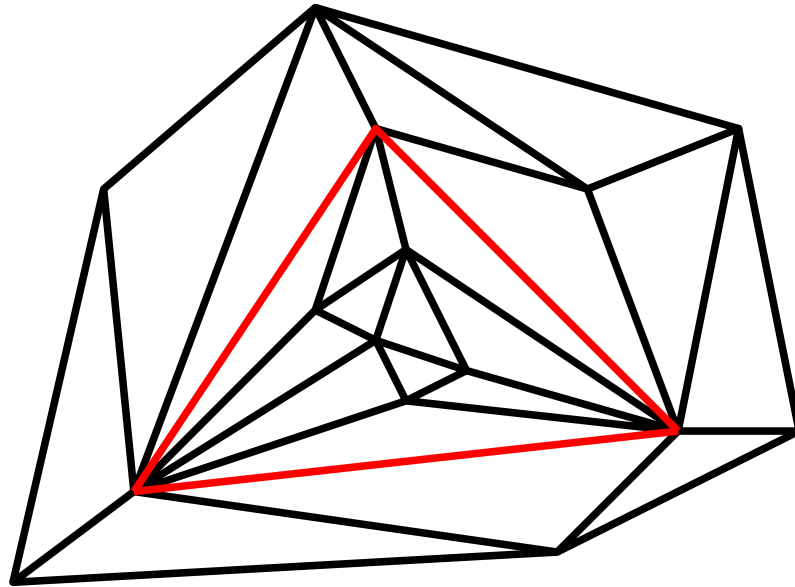
- ▶ The *independence complex* $\text{Ind}(H)$ has to be a sphere.
- ▶ The *clique complex* $\text{Clique}(\overline{H})$ has to be a sphere.

We have:

- ▶ $\text{SK}_1(\text{Clique}(\overline{H})) = \overline{H}$.
 - ▶ Every clique in $\text{SK}_1(\text{Clique}(\overline{H}))$ forms a face of $\text{Clique}(\overline{H})$.
-

flag simplicial complexes

A simplicial complex K is *flag*,
if every clique in its 1-skeleton $SK_1(K)$ spans a simplex,
i.e., no “empty simplices”.



THEOREM

CSORBA & L. (2003)

$\text{Hom}(H, K_n)$ is a PL manifold for all $n \geq \chi(H)$
iff H is the complement of the 1-skeleton
of a flag simplicial PL sphere.

THEOREM

CSORBA & L. (2003)

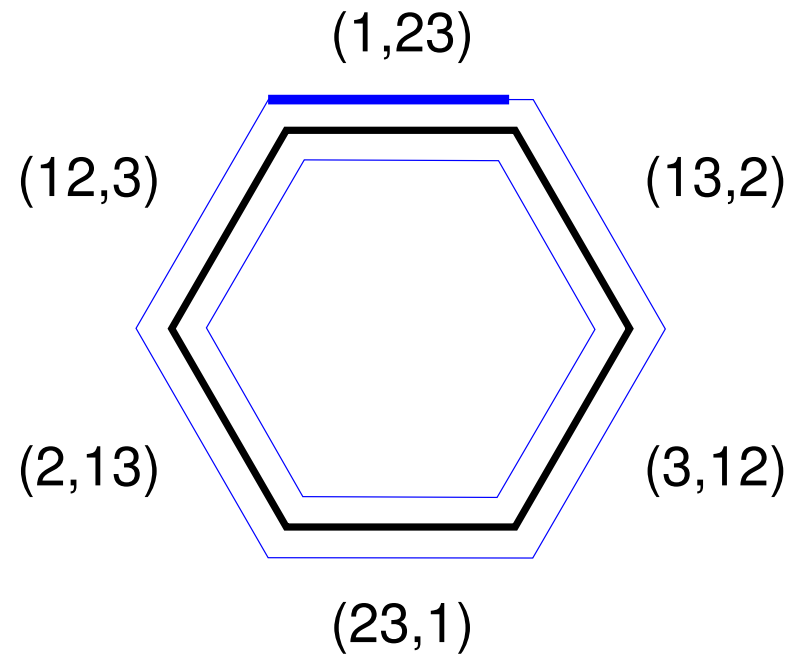
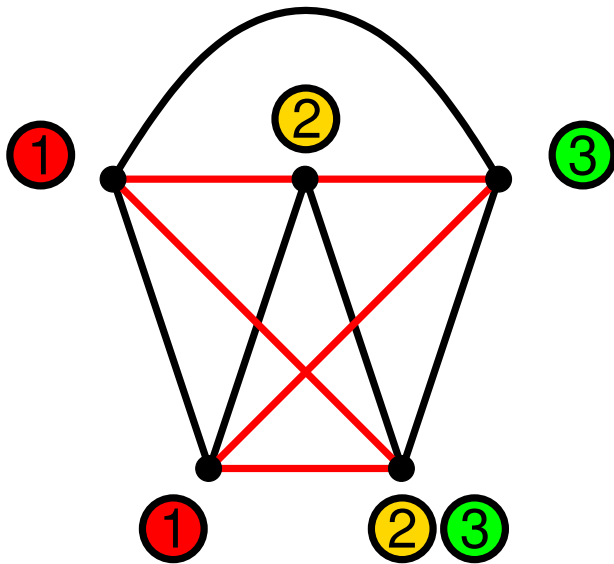
$\text{Hom}(H, K_n)$ is a PL manifold for all $n \geq \chi(H)$
iff H is the complement of the 1-skeleton
of a flag simplicial PL sphere.

For such graphs H , we call $\text{Hom}(H, K_n)$
a *graph coloring manifold*.

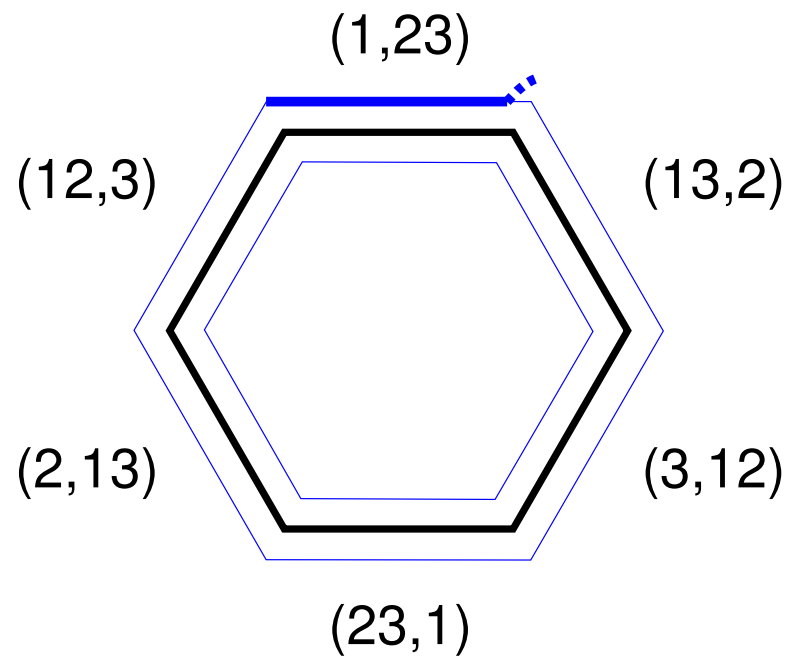
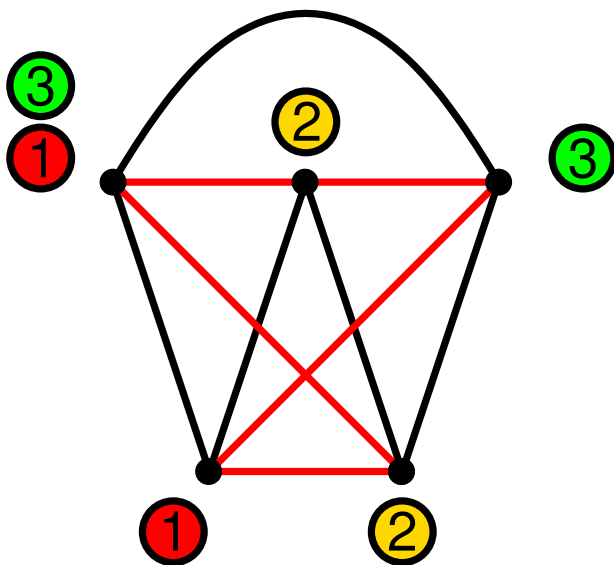
6. One-dimensional flag spheres

$$\mathrm{Hom}(\overline{C}_5, K_3) = S^0 \times S^1$$

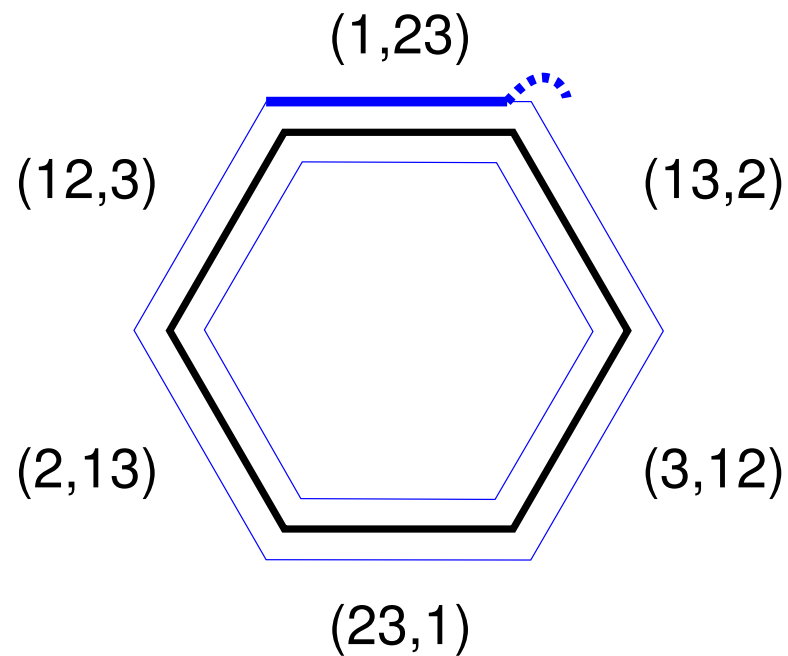
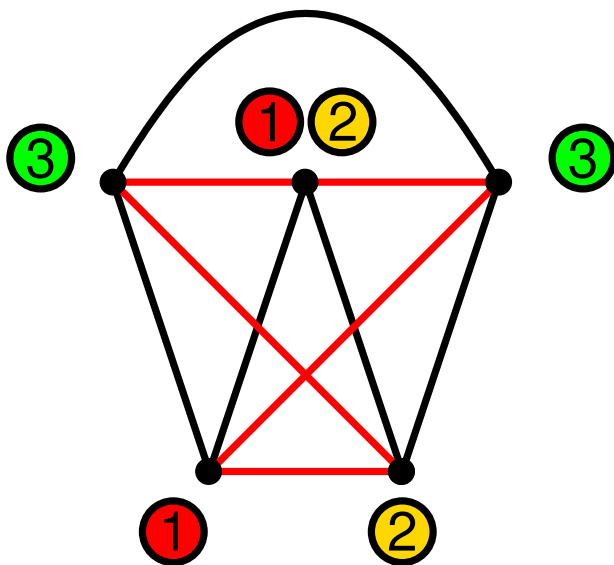
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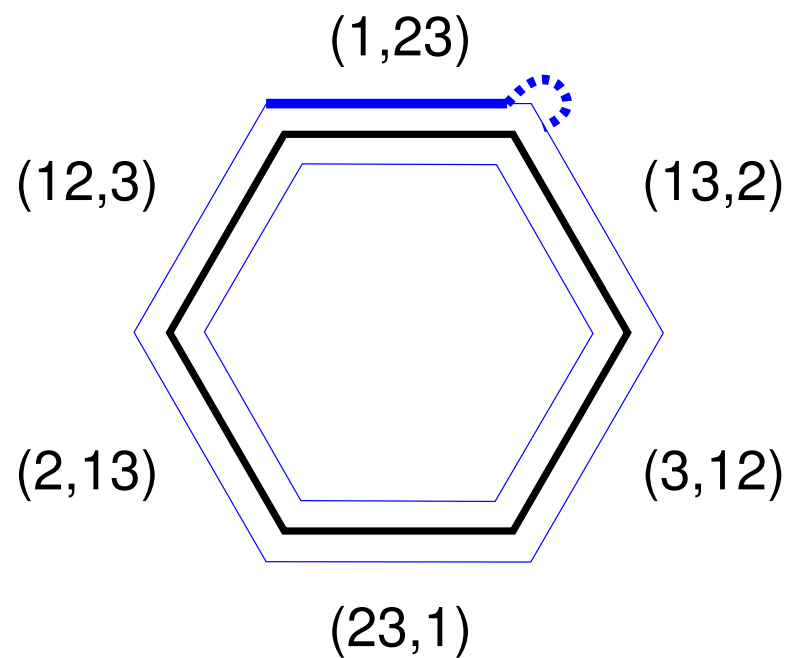
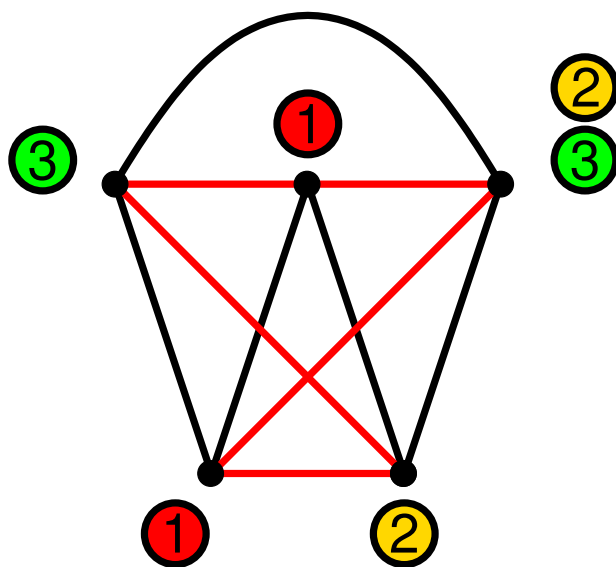
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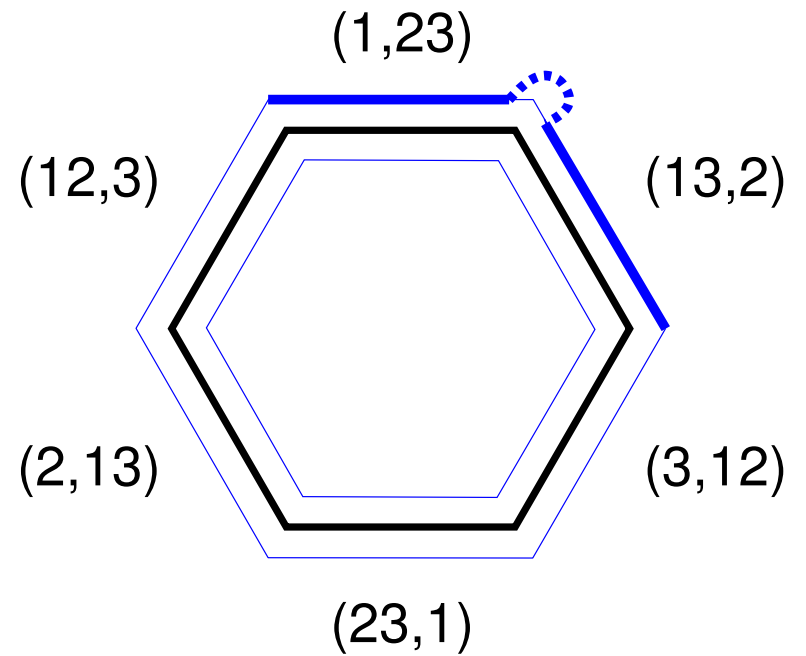
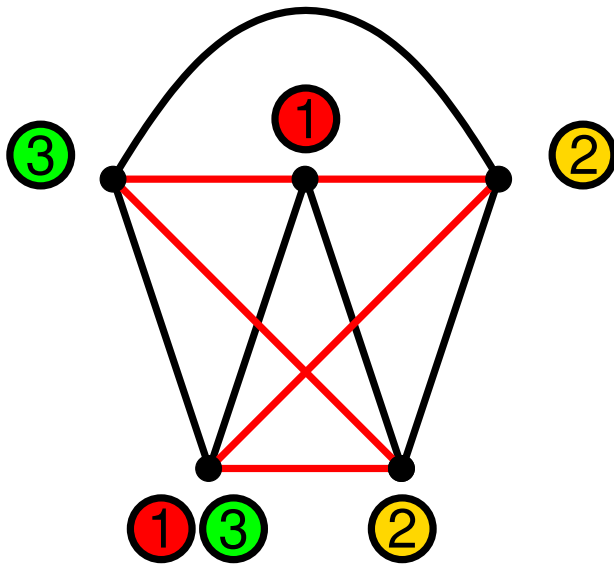
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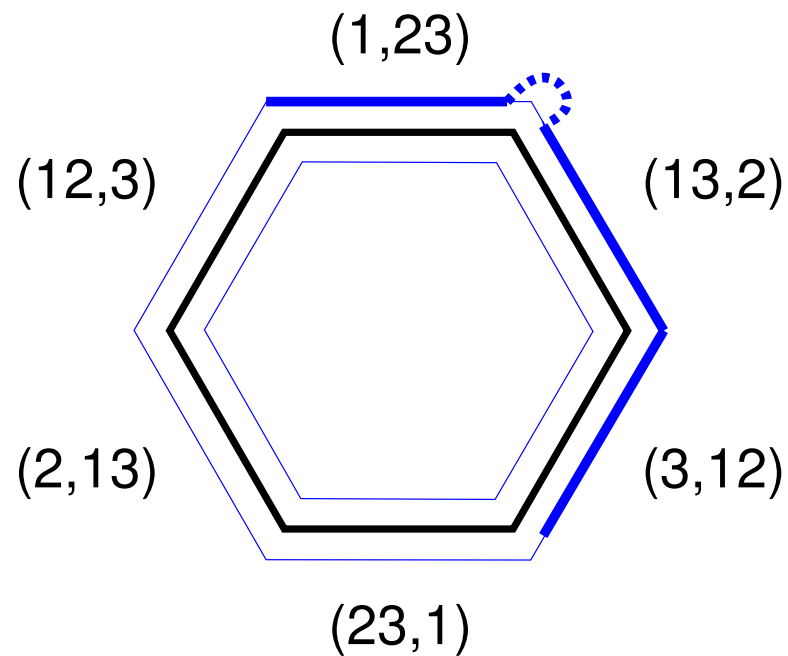
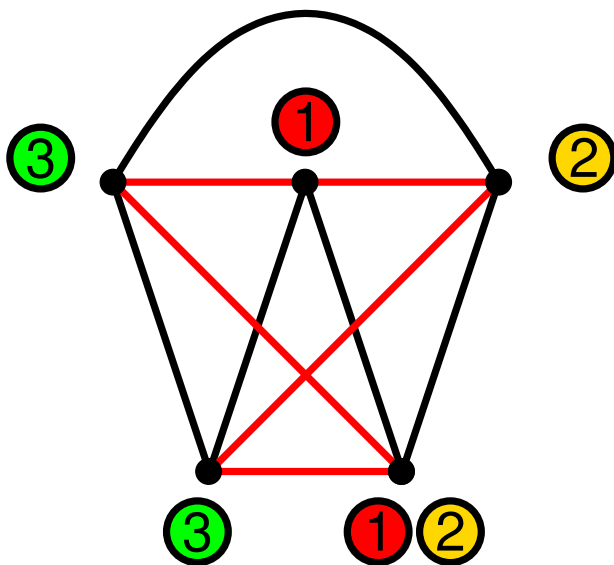
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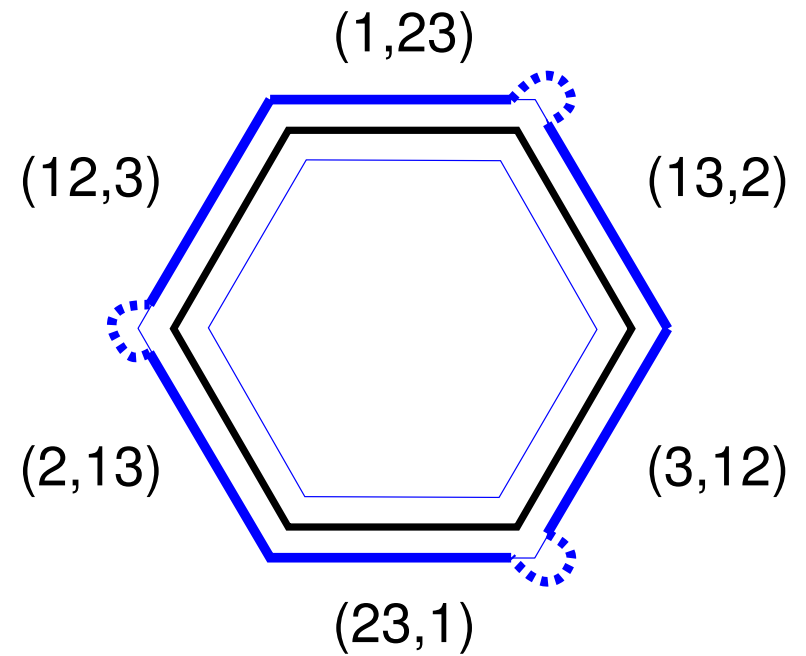
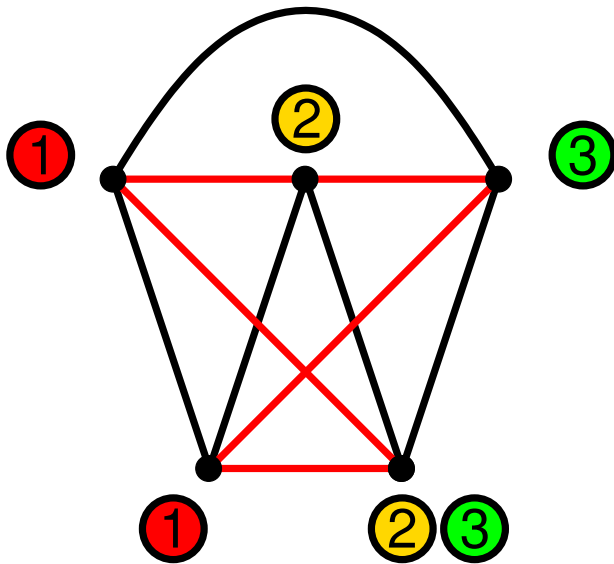
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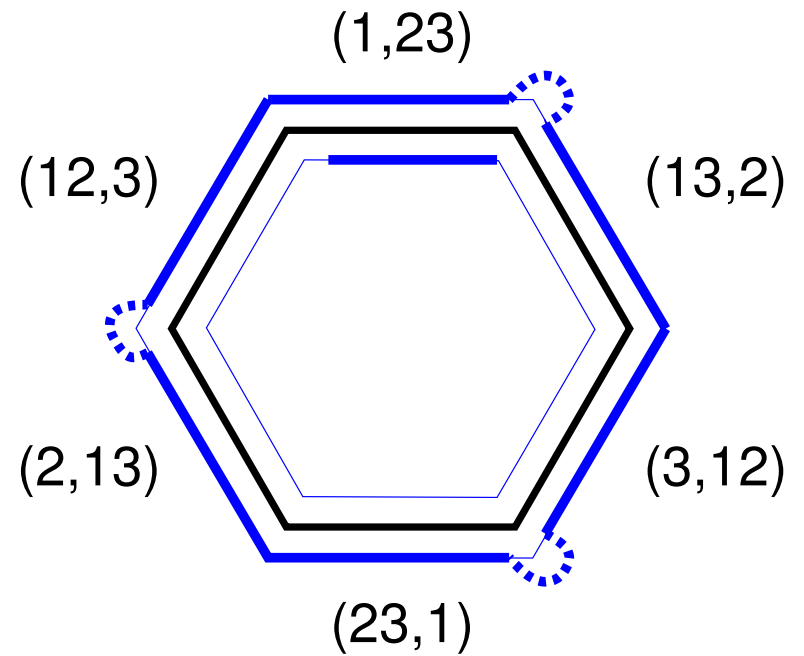
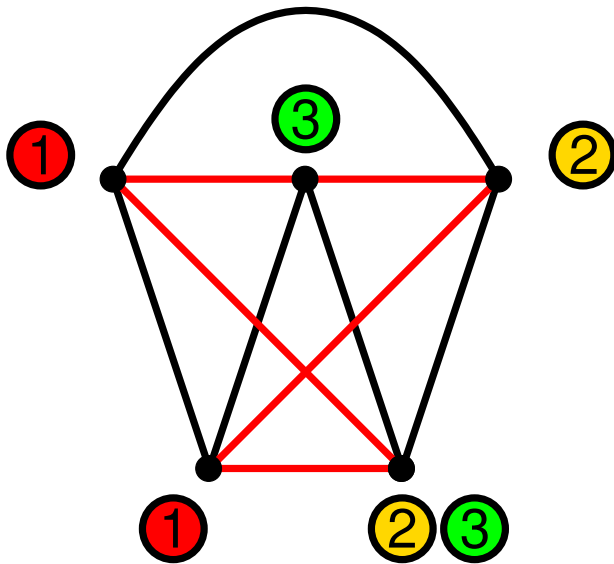
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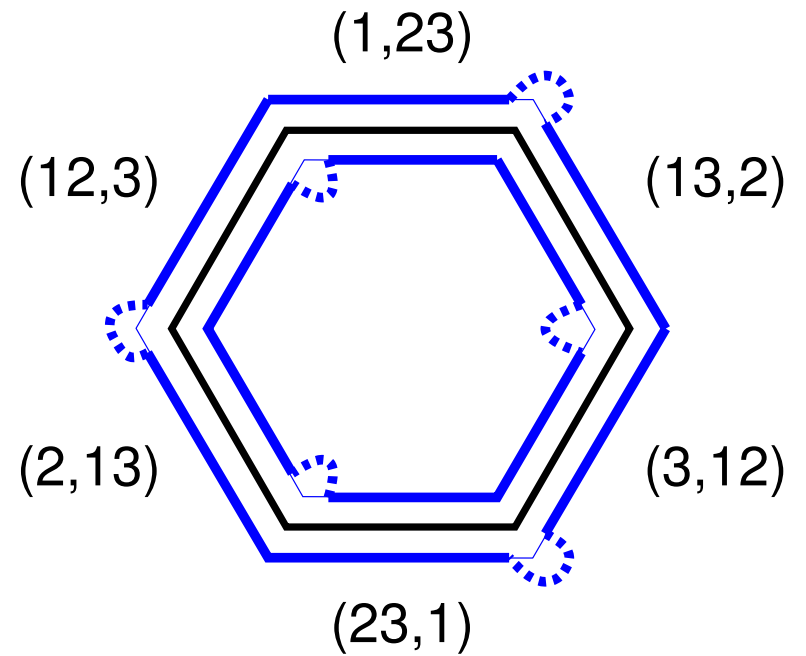
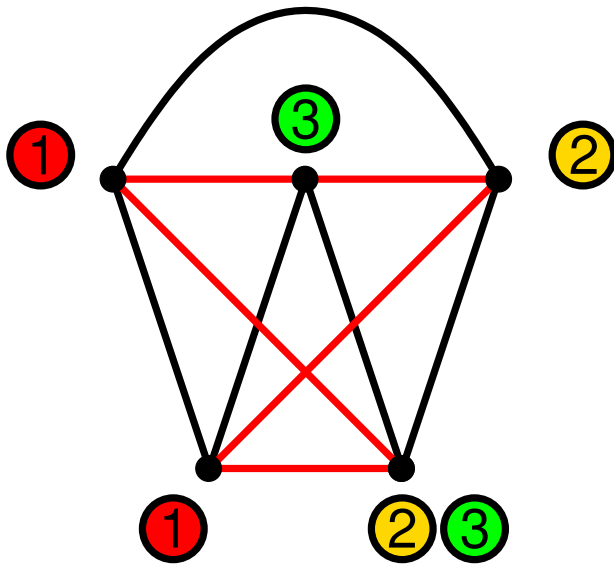
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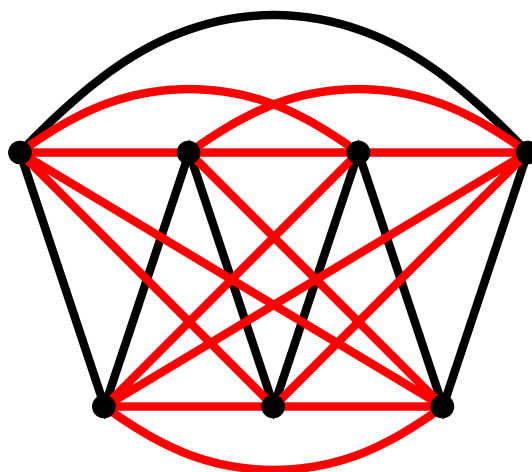
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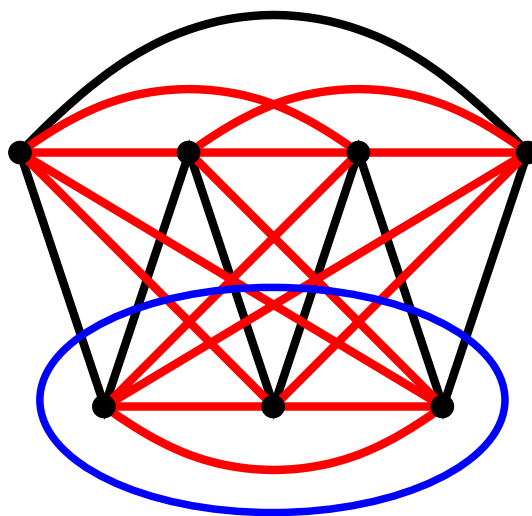
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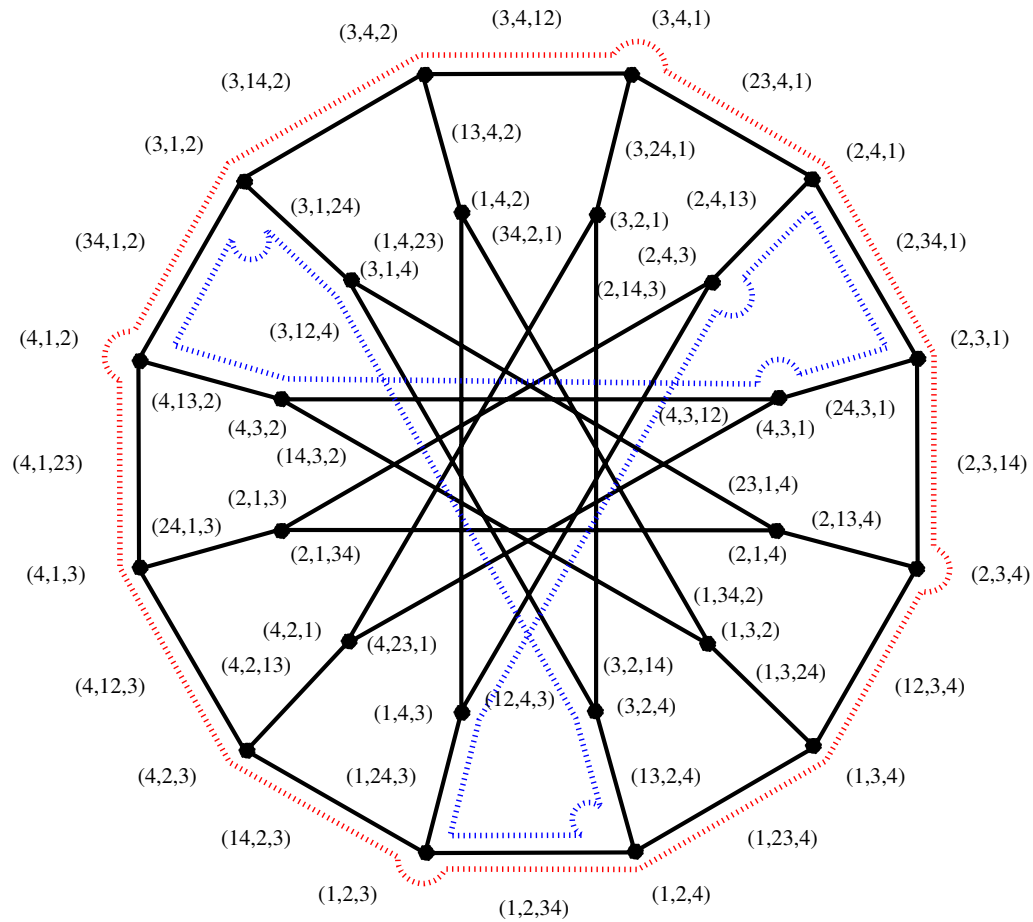
C_7 and $\overline{C}_7$:



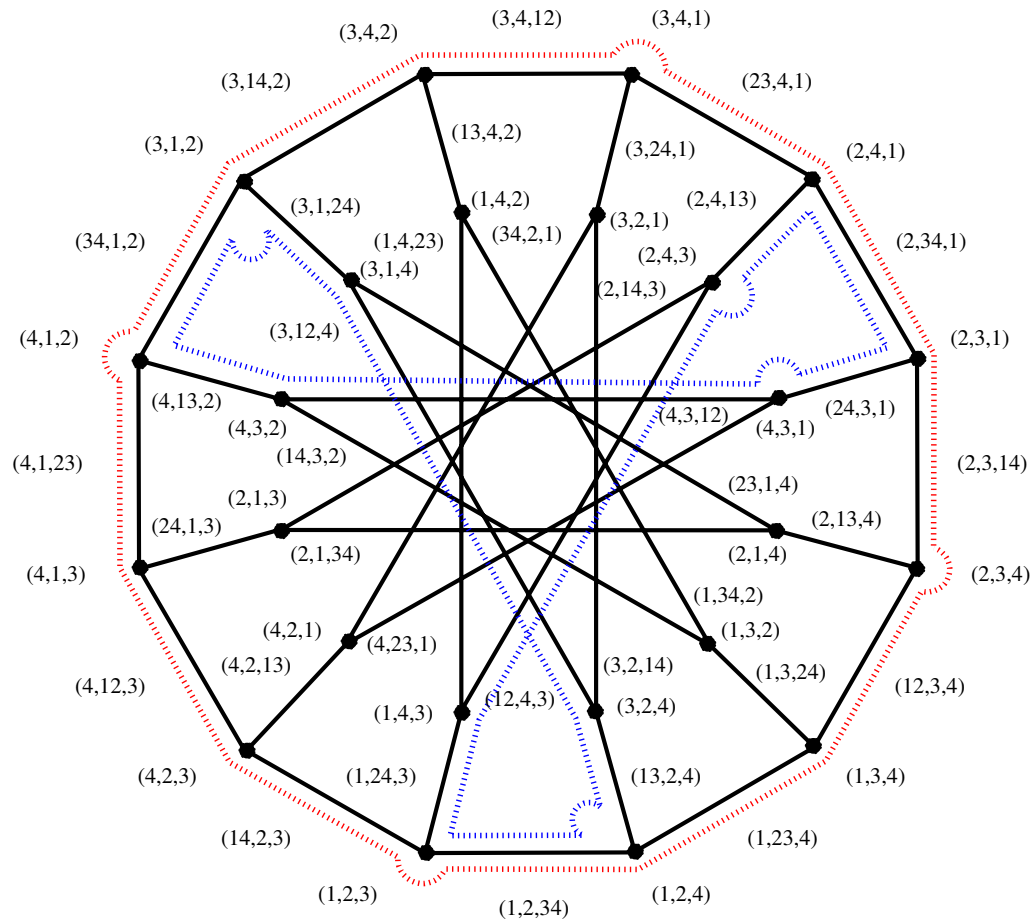
C_7 and $\overline{C}_7$:



$\text{Hom}(K_3, K_4)$ and $\text{Hom}(\overline{C}_7, K_4)$:



$\text{Hom}(K_3, K_4)$ and $\text{Hom}(\overline{C}_7, K_4)$:



$\text{Hom}(K_r, K_{r+1})$ is a graph with $(r + 1)!$ vertices and $\frac{r}{2}(r + 1)!$ edges.

PROPOSITION

CSORBA & L. (2003)

$\text{Hom}(\overline{C}_{2r+1}, K_{r+1})$ is the disjoint union of $r!$ circles
with $(2r^2 + 3r + 1)$ vertices each.

PROPOSITION

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$\text{Hom}(\overline{C}_{2r+1}, K_{r+1})$ is the disjoint union of $r!$ circles with $(2r^2 + 3r + 1)$ vertices each.

CONJECTURE

L. (2003)

$$H_*(\text{Hom}(\overline{C}_{2r+1}, K_{r+2})) = (\mathbb{Z}, \mathbb{Z}^{2(r!-2)} \oplus \mathbb{Z}_{r!}, \mathbb{Z}^{2(r!-2)}, \mathbb{Z}).$$

PROPOSITION

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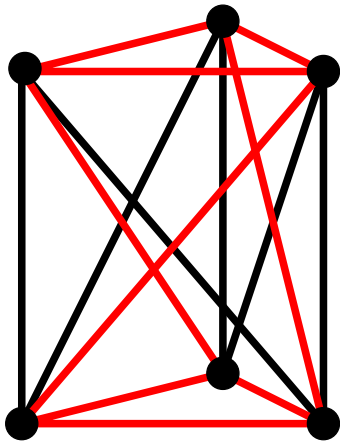
$$H_*(\text{Hom}(\overline{C}_{2r+1}, K_{r+2})) = (\mathbb{Z}, \mathbb{Z}^{2(r!-2)} \oplus \mathbb{Z}_{r!}, \mathbb{Z}^{2(r!-2)}, \mathbb{Z}).$$

$$H_*(\text{Hom}(\overline{C}_5, K_4)) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}).$$

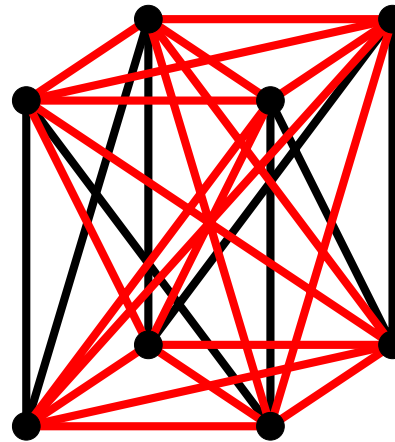
$$H_*(\text{Hom}(\overline{C}_7, K_5)) = (\mathbb{Z}, \mathbb{Z}^8 \oplus \mathbb{Z}_6, \mathbb{Z}^8, \mathbb{Z}).$$

complements of even circles

C_6 and $\overline{C}_6$:

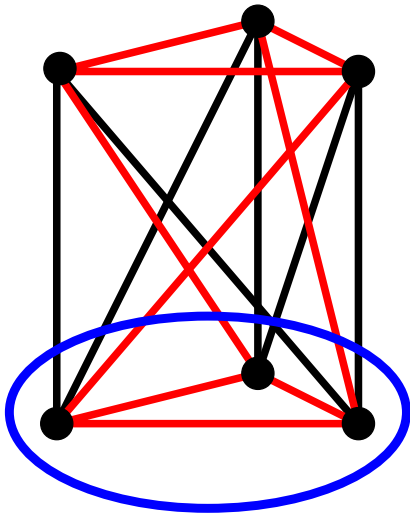


C_8 and $\overline{C}_8$:

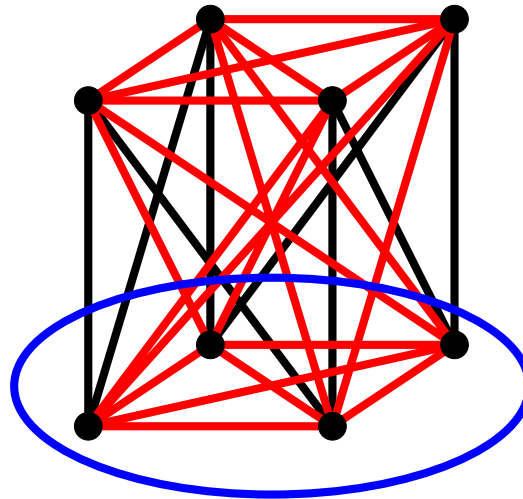


complements of even circles

C_6 and $\overline{C}_6$:



C_8 and $\overline{C}_8$:



THEOREM

BABSON & KOZLOV (2003)

$$\text{Hom}(K_s, K_r) \simeq \bigvee_{f(s,r)} S^{r-s}.$$

THEOREM

BABSON & KOZLOV (2003)

$$\text{Hom}(K_s, K_r) \simeq \bigvee_{f(s,r)} S^{r-s}.$$

PROPOSITION

BABSON & KOZLOV (2003)

The numbers $f(s, r)$ satisfy the recurrence relation

$$f(s, r) = sf(s-1, r-1) + (s-1)f(s, r-1),$$

for $r > s \geq 2$. $f(r, r) = r! - 1$, $f(1, r) = 0$ for $r \geq 1$,
and $f(s, r) = 0$ for $s > r$.

PROPOSITION

CSORBA & L. (2004)

$\text{Hom}(\overline{C}_{2r}, K_{r+1})$ is an orientable cubical surface of
of genus

$$g(r) = f(r, r+1) = \binom{r}{2} r! - \sum_{m=1}^{r-1} m(m+2)(m+3) \cdots r$$

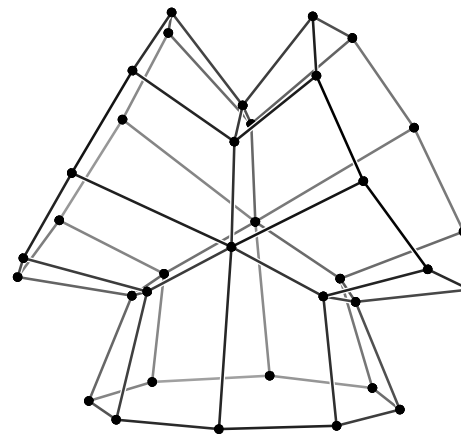
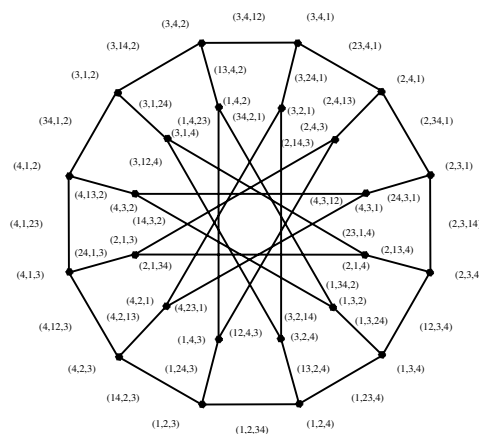
with

$n(r) := (2 + r^2) \cdot (r+1)!$ vertices,

$2(n(r) + 2g(r) - 2)$ edges,

$n(r) + 2g(r) - 2$ squares.

complements of even circles C_{2s}



$$\begin{aligned}g(2) &= 1, \\g(3) &= 13, \\g(4) &= 121, \\g(5) &= 1081, \\g(6) &= 10081, \\g(7) &= 100801, \\g(8) &= 1088641, \\g(9) &= 12700801, \\g(10) &= 159667201.\end{aligned}$$

CONJECTURE

L. (2004)

$\text{Hom}(\overline{C}_{2s}, K_r)$ is homeomorphic to the connected sum of $2f(s, r)$ copies of $S^{r-s} \times S^{r-s}$ for $s \geq 2$.

CONJECTURE

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THEOREM

CSORBA & L. (2004)

The local pieces are homeomorphic to $B^{r-s} \times S^{r-s}$.

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THEOREM

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The local pieces are homeomorphic to $B^{r-s} \times S^{r-s}$.

$$H_*(\text{Hom}(\overline{C}_6, K_4)) = (\mathbb{Z}, \mathbb{Z}^{26}, \mathbb{Z}).$$

$$H_*(\text{Hom}(\overline{C}_6, K_5)) = (\mathbb{Z}, 0, \mathbb{Z}^{58}, 0, \mathbb{Z}).$$

CONJECTURE

L. (2004)

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$$H_*(\text{Hom}(\overline{C}_6, K_5)) = (\mathbb{Z}, 0, \mathbb{Z}^{58}, 0, \mathbb{Z}).$$

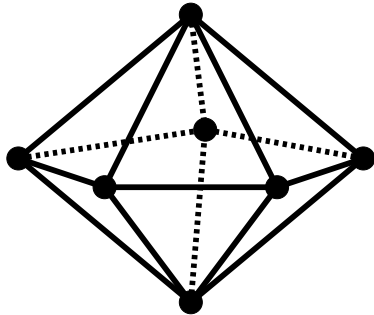
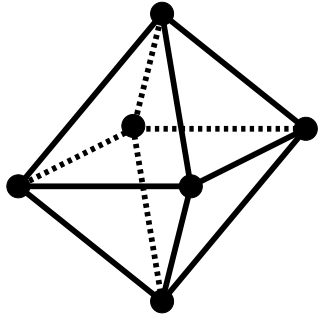
(If $\text{Hom}(\overline{C}_6, G)$ is k -connected, then $\chi(G) \geq k + 4$.)

7. Two-dimensional flag spheres

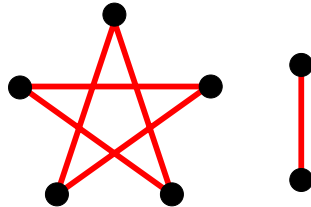
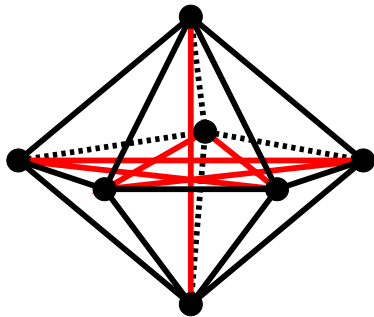
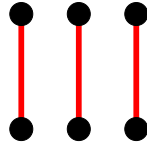
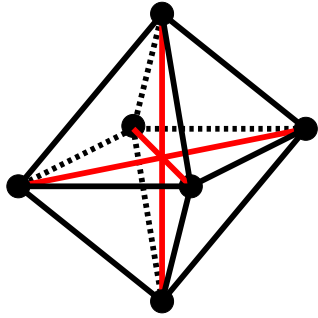
two-dimensional flag spheres

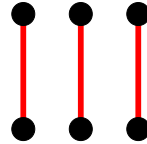
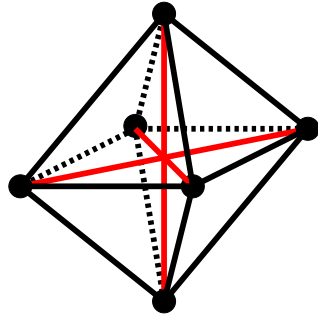
# vertices	6	7	8	9	10
# manifolds:	3	9	43	655	42426
# spheres:	2	5	14	50	233
# flag spheres	1	1	2	4	10

flag 2-spheres ($n = 6, 7$)

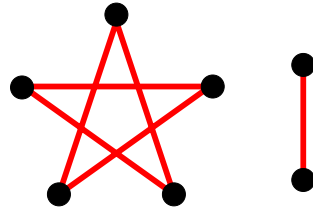
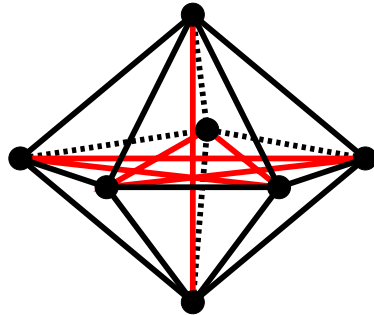


flag 2-spheres ($n = 6, 7$)



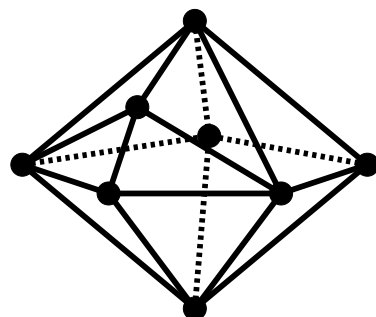
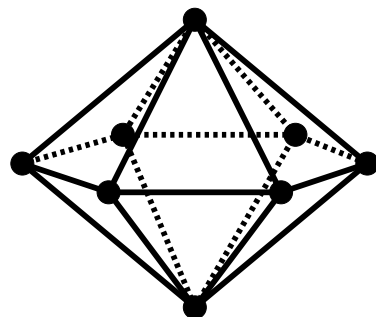


$$\text{Hom}(\overline{\text{SK}_1(\partial C_3^\Delta)}, K_r) = \text{Hom}(K_2, K_r) \times \text{Hom}(K_2, K_r) \times \text{Hom}(K_2, K_r)$$

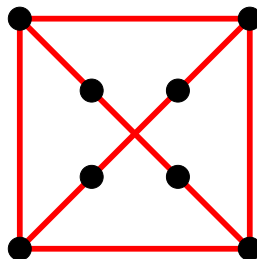
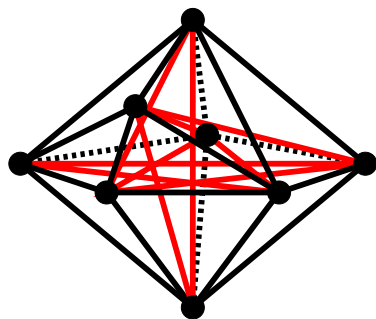
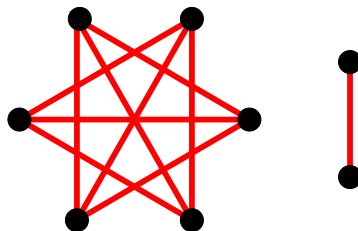
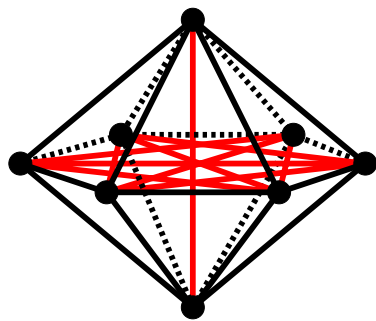


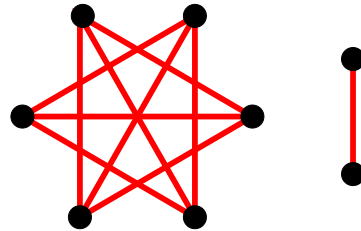
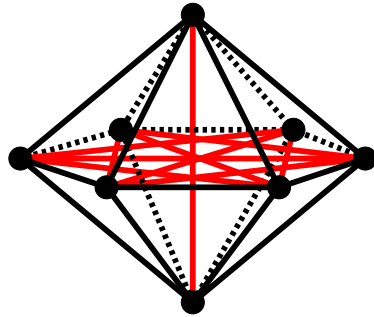
$$\text{Hom}(\overline{\text{SK}_1(C_5 * S^0)}, K_r) = \text{Hom}(\overline{C_5}, K_r) \times \text{Hom}(K_2, K_r)$$

flag 2-spheres ($n = 8$)

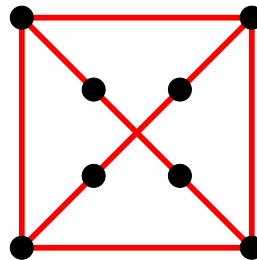
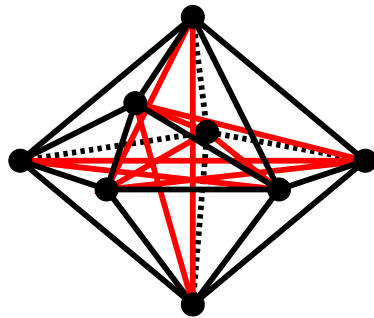


flag 2-spheres ($n = 8$)





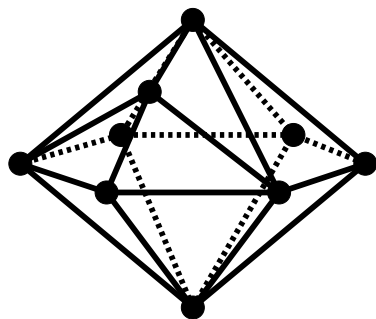
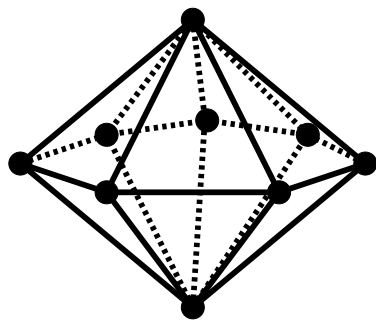
$$\text{Hom}(\overline{\text{SK}_1(C_6 * S^0)}, K_r) = \text{Hom}(\overline{C_6}, K_r) \times \text{Hom}(K_2, K_r)$$



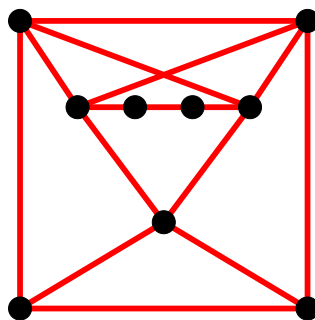
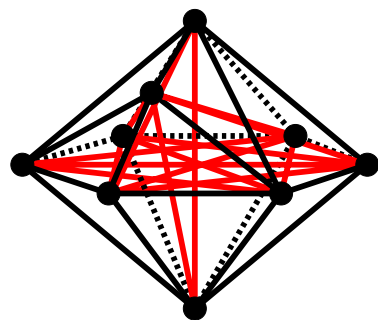
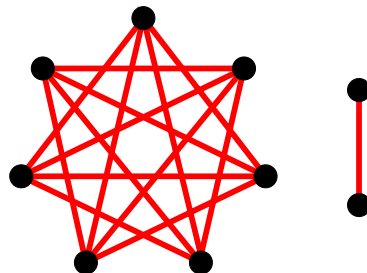
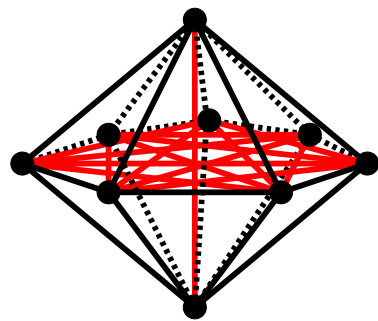
$$\text{Hom}(\overline{\text{SK}_1(^2 8_{43})}, K_3): \quad 96 \text{ vertices, 4 circles}$$

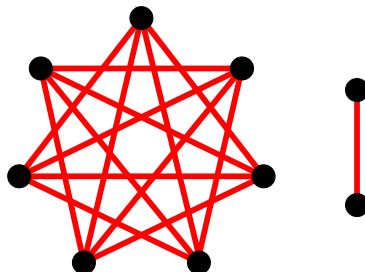
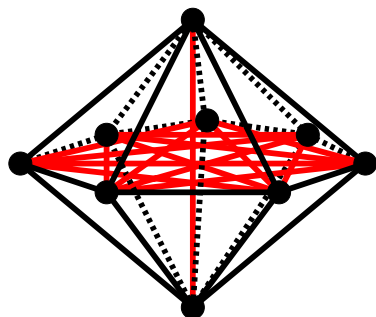
$$\text{Hom}(\overline{\text{SK}_1(^2 8_{43})}, K_4): \quad 3624 \text{ vertices, } H_* = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z})$$

flag 2-spheres ($n = 9$)

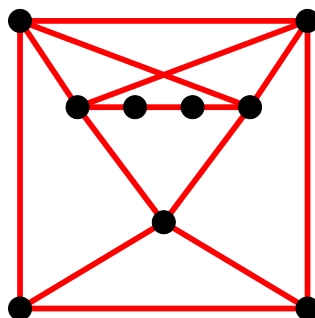
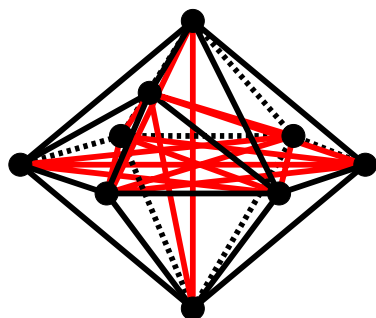


flag 2-spheres ($n = 9$)





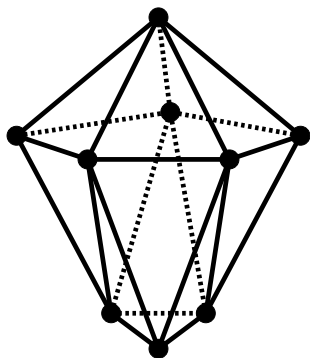
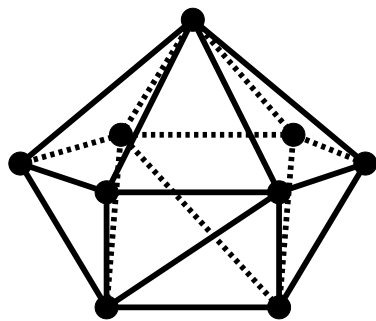
$$\text{Hom}(\overline{\text{SK}_1(C_7 * S^0)}, K_r) = \text{Hom}(\overline{C_7}, K_r) \times \text{Hom}(K_2, K_r)$$



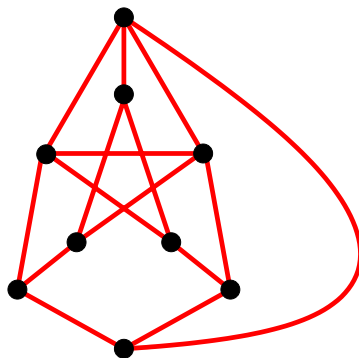
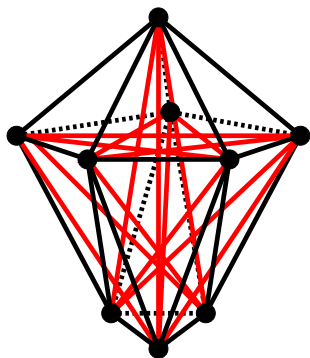
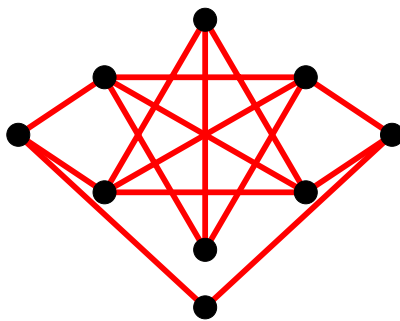
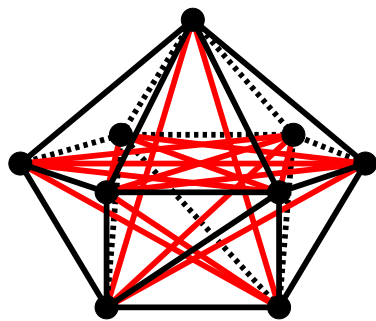
$$\text{Hom}(\overline{\text{SK}_1(^29_{651})}, K_3): \quad 24 \text{ vertices}$$

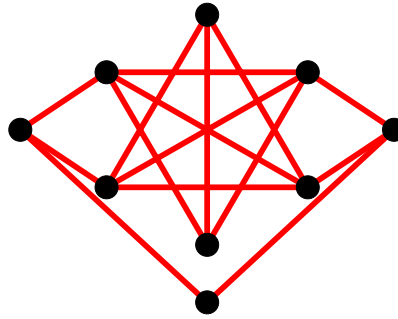
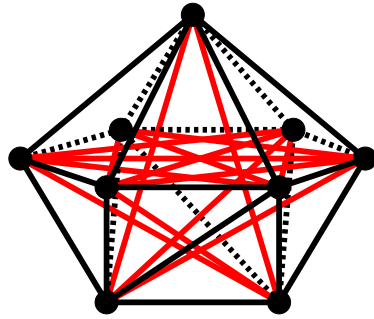
$$\text{Hom}(\overline{\text{SK}_1(^29_{651})}, K_4): \quad 2928 \text{ vertices, } H_* = (\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$$

flag 2-spheres ($n = 9$)



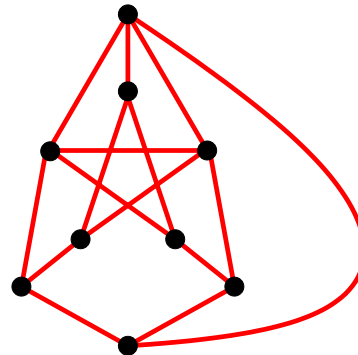
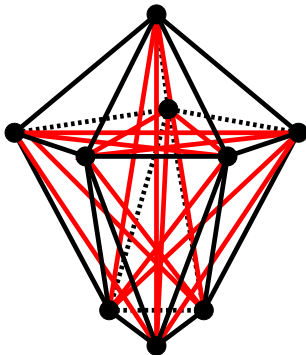
flag 2-spheres ($n = 9$)





$\text{Hom}(\overline{\text{SK}_1(29_{652})}, K_3)$: 24 vertices

$\text{Hom}(\overline{\text{SK}_1(29_{652})}, K_4)$: 3120 vertices, $H_* = (\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$



$\text{Hom}(\overline{\text{SK}_1(29_{655})}, K_3)$: 12 vertices

$\text{Hom}(\overline{\text{SK}_1(29_{655})}, K_4)$: 3096 vertices, $H_* = (\mathbb{Z}, \mathbb{Z}^{13}, \mathbb{Z}^{13}, \mathbb{Z})$

Let S_n^d be a flag simplicial d -sphere with n vertices.

QUESTION

What are good candidates for S_n^d such that $\text{Hom}(\overline{\text{SK}_1(S_n^d)}, K_r)$ is highly connected?

Let S_n^d be a flag simplicial d -sphere with n vertices.

QUESTION

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QUESTION

General properties of graph coloring manifolds?