Topological Obstructions to Graph Colorings

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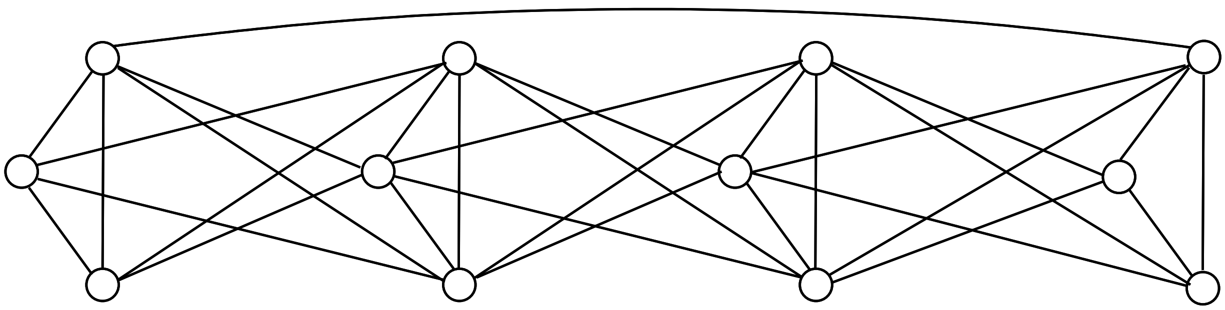
Fragestellung: How many colors does one need to color the vertices of a given graph G, so that if two vertices are connected by an edge, then they get different colors?



The minimal number of colors is called **chromatic number** and is denoted by $\boldsymbol{\chi(G)}$.

In general, it is very difficult to compute $\chi(G)$ or even to find bounds.

It is even NP-complete to decide whether $\chi(G) = 3$.

Ansatz:

$$\begin{array}{ccc} \text{Graph } G & \longrightarrow & \text{Topological space } X(G) \\ & & \downarrow \\ \begin{array}{c}\text{Combinatorial}\\ \text{properties of } G\end{array} & \longleftarrow & \begin{array}{c}\text{Topological}\\ \text{properties of } X(G)\end{array} \end{array}$$

The topological spaces occurring in this context are finite **regular CW complexes**, where the cells are products of simplices.

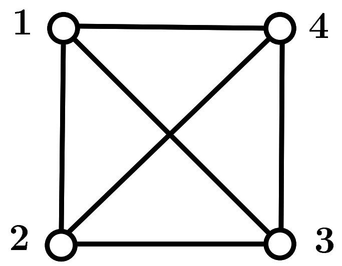
At our disposal we have theorems expressing *topological obstructions*, such as the Borsuk-Ulam theorem.

Definition. Let G be a graph. The **neighborhood complex** $\boldsymbol{\mathcal{N}(G)}$ is a simplicial complex defined as follows:

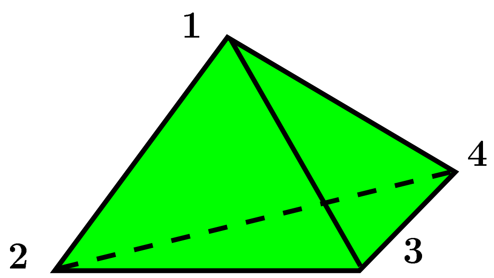
- vertices of $\mathcal{N}(G)$ are the vertices of G;
- a set of vertices A forms a simplex if and only if all vertices in A have a common neighbor.

In particular, every vertex v in G defines a maximal simplex consisting of all neighbors of v.

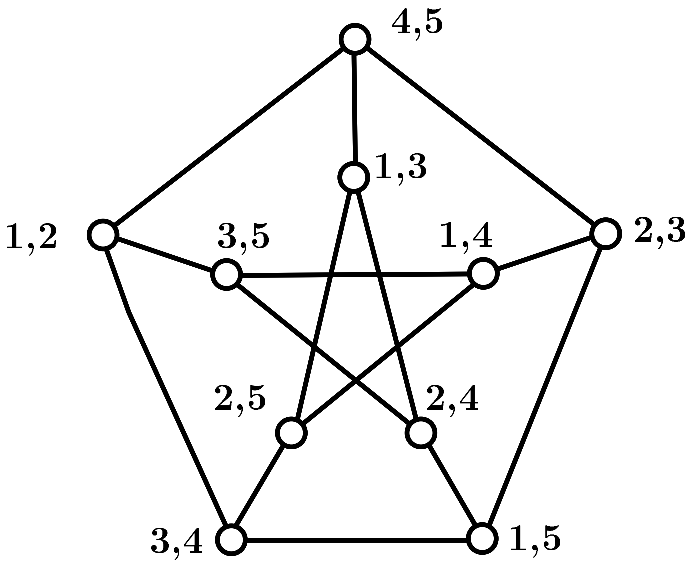
Examples.



$\boldsymbol{K_4}$



$\boldsymbol{\mathcal{N}(K_4)}$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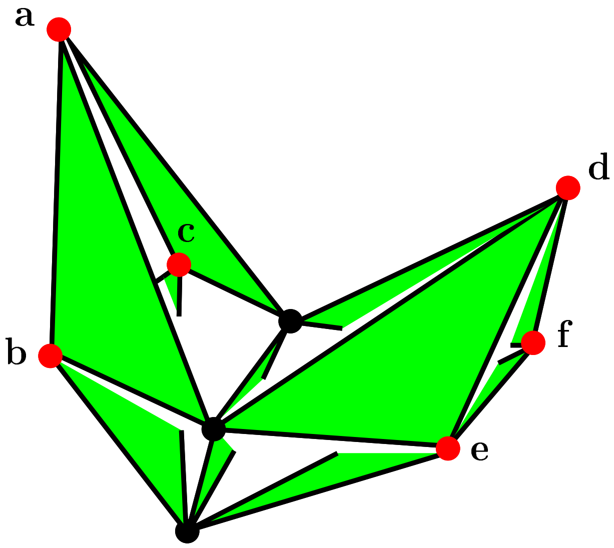
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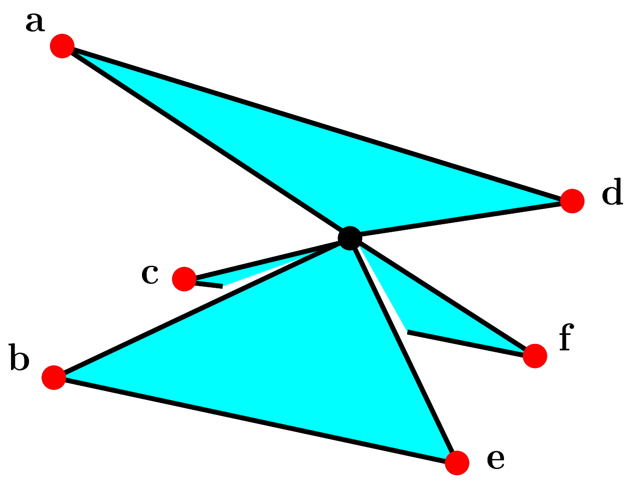
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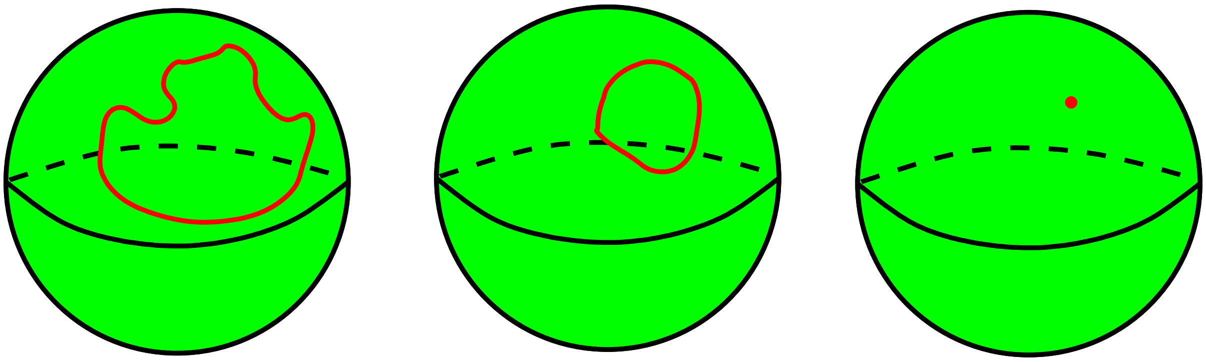
Definition. A topological space C is called ***k*-connected**, if every map $\phi : S^k \to C$ can be extended to a map $\widetilde{\phi} : B^{k+1} \to C$.

Equivalently: **homotopy groups** up to dimension k are equal 0.



Theorem (Lovász, 1978).

Let G be a graph and $k \in \mathbb{Z}$, $k \geq -1$.
Then

$$\mathcal{N}(G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+3\,.$$

Using this theorem Lovász has proved the Kneser Conjecture.

Equivalently, Lovász' theorem can be formulated as follows:

$$\mathtt{Hom}\,(K_2, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+3\,.$$

Lovász Conjecture.

Let G be a graph and $r, k \in \mathbb{Z}$, $r \geq 1$, $k \geq -1$.
Then

$$\mathtt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+4\,.$$

Theorem (Babson & K., 2003).

(a) The Lovász Conjecture is true.

(b) Let $m, k \in \mathbb{Z}$, $m \geq 1$, $k \geq -1$. Then

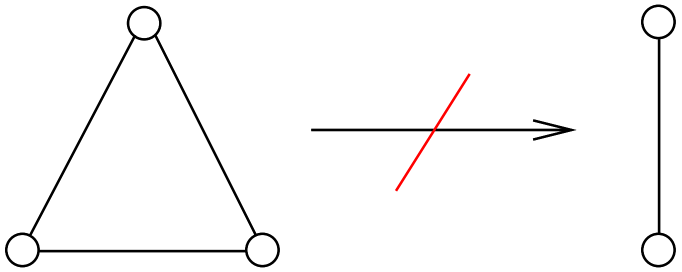
$$\mathtt{Hom}\,(K_m, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+m+1\,.$$

Definition.

Let T, G be two graphs. A **homomorphism** from T to G is a map $\phi : V(T) \to V(G)$, such that for every edge (x, y) in T the image $(\phi(x), \phi(y))$ is an edge in G.

Observation:

$$G \text{ is } n\text{-colorable} \iff \exists\, \phi : G \to K_n.$$



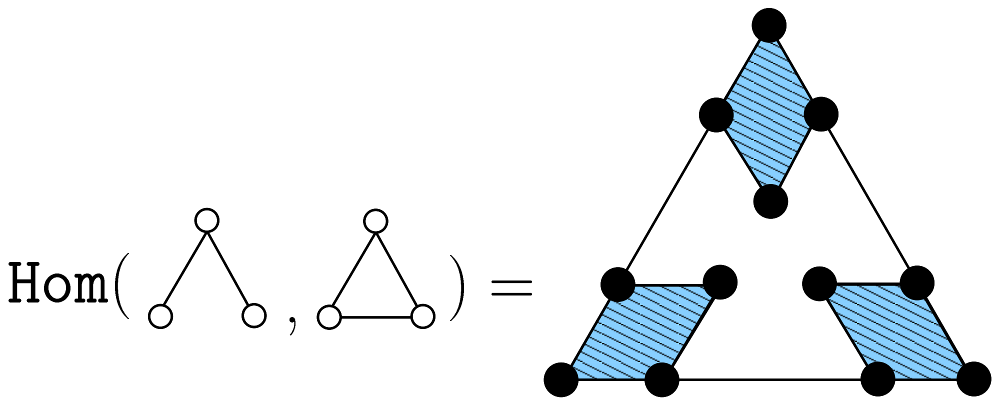
A composition of two homomorphisms $\phi_1 : G_1 \to G_2$ and $\phi_2 : G_2 \to G_3$ is again a homomorphism $\phi_2 \circ \phi_1 : G_1 \to G_3$.

This gives **Graphs** - the category which has graphs as objects and graph homomorphisms as morphisms.

Idea.

- The vertices of G are the colors.
- A graph homomorphism is a valid coloring.
- A cell in $\texttt{Hom}\,(T, G)$ is a collection of color lists, one for each vertex of T, such that an arbitrary choice of colors, one from each list, is a valid coloring of T.

Example.



Definition.

Let T and G be two graphs. The $\mathtt{Hom}$-complex $\mathtt{Hom}\,(T, G)$ is a regular CW complex defined as follows:

- cells are all the maps $\eta : V(T) \to 2^{V(G)} \setminus \{\emptyset\}$, such that for any $(x, y) \in E(T)$, $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$, we have $(\tilde{x}, \tilde{y}) \in E(G)$;

- the closure of each cell η consists of all $\tilde{\eta} : V(T) \to 2^{V(G)} \setminus \{\emptyset\}$, such that $\tilde{\eta}(v) \subseteq \eta(v)$, for all $v \in V(T)$.

Properties of the $\texttt{Hom}$-complexes:

(1) Cells in $\texttt{Hom}\,(T, G)$ are products of simplices. More precisely, every cell η is a product of $|V(T)|$ simplices of dimension $|\eta(x)| - 1$ for $x \in V(T)$.

(2) $\texttt{Hom}\,(K_2, G)$ is homotopy equivalent to $\mathcal{N}(G)$.

(3) $\texttt{Hom}\,(T, -)$ is a covariant and $\texttt{Hom}\,(-, G)$ is a contravariant functor from **Graphs** to **Top**.

→ If $\phi : G \to G'$ is a graph homomorphism, then, for an arbitrary graph H, we shall denote the induced cellular maps as
$\phi^H : \texttt{Hom}\,(H, G) \to \texttt{Hom}\,(H, G')$ and
$\phi_H : \texttt{Hom}\,(G', H) \to \texttt{Hom}\,(G, H)$.

→ As a consequence of (3), the group $\mathcal{A}ut(T) \times \mathcal{A}ut(G)$ acts on $\texttt{Hom}\,(T, G)$.

→ When G has no loops and $\phi \in \mathcal{A}ut(T)$ flips an edge, the induced map $\phi_G \circlearrowright \texttt{Hom}\,(T, G)$ has no fixed points.

Definition.

A topological space X is called **$\mathbb{Z}_2$-space**, if $\mathbb{Z}_2$ acts freely on X.

In this case, there exists a continuous map $\psi : X \to S^\infty$. The induced quotient map $\phi : X/\mathbb{Z}_2 \to \mathbb{RP}^\infty$ is up to homotopy independent of the choice of ψ.

It induces an algebra map

$$\phi^* : H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2),$$

which is independent of the choice of ψ.

Let z be the generator of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Then

$$\boldsymbol{w_1(X)} := \phi^*(z)$$

is called the **Stiefel-Whitney class.**

Stiefel-Whitney classes are functorial:

if $\psi : X \to Y$ is a $\mathbb{Z}_2$-map, and $\widetilde{\psi} : X/\mathbb{Z}_2 \to Y/\mathbb{Z}_2$ is the induced map between the quotient spaces, then

$$\widetilde{\psi}^*(w_1(Y)) = w_1(X)\,.$$

An application of Stiefel-Whitney classes:

Borsuk-Ulam Theorem.

There is no continuous map $S^{k+1} \longrightarrow S^k$, which commutes with the antipodal maps on S^{k+1} and S^k.

Lovász Conjecture:

$$\texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad k\text{-connected} \quad \Longrightarrow \quad \chi(G) \geq k+4\,.$$

$$k=-1: \quad \texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad \text{nonempty} \quad \Longrightarrow \quad \chi(G) \geq 3\,.$$

$$k=0: \quad \texttt{Hom}\,(C_{2r+1}, G) \quad \text{is} \quad \text{connected} \quad \Longrightarrow \quad \chi(G) \geq 4\,.$$

Proof for $k=0$.
Assume $\chi(G) \leq 3$, then there exists a graph homomorphism

$$\phi : G \to K_3.$$

It induces a $\mathbb{Z}_2$-map

$$\phi^{C_{2r+1}} : \texttt{Hom}\,(C_{2r+1}, G) \to \texttt{Hom}\,(C_{2r+1}, K_3).$$

A direct analysis shows that the connected components of $\texttt{Hom}\,(C_{2r+1}, K_3)$ can be indexed with the number of times C_{2r+1} winds around K_3. This number α is odd:

$$\alpha = \pm 1, \pm 3, \ldots, \pm(2s+1),$$

and $s \geq 0$. The $\mathbb{Z}_2$-action on $\texttt{Hom}\,(C_{2r+1}, K_3)$ swaps the connected components by changing the sign of the winding number.
This contradicts the fact that $\texttt{Hom}\,(C_{2r+1}, G)$ is connected.

Scheme of the Proof (Babson & K., 2003):

• Since $\texttt{Hom}\,(C_{2r+1}, G)$ is k-connected there exists a $\mathbb{Z}_2$-map

$$f : S_a^{k+1} \to \texttt{Hom}\,(C_{2r+1}, G).$$

• Assume $\chi(G) \leq k+3$, then $\exists\, \phi : G \to K_{k+3}$.
This induces a $\mathbb{Z}_2$-map

$$\phi^{C_{2r+1}} : \texttt{Hom}\,(C_{2r+1}, G) \to \texttt{Hom}\,(C_{2r+1}, K_{k+3}).$$

• Since $\phi^{C_{2r+1}} \circ f : S_a^{k+1} \to \texttt{Hom}\,(C_{2r+1}, K_{k+3})$ is a $\mathbb{Z}_2$-map and $w_1^{k+1}(S_a^{k+1}) \neq 0$, we conclude that

$$w_1^{k+1}(\texttt{Hom}\,(C_{2r+1}, K_{k+3})) \neq 0.$$

• On the other hand, for even k, involved spectral sequence computations yield $w_1^{k+1}(\texttt{Hom}\,(C_{2r+1}, K_{k+3})) = 0 \Longrightarrow$ a contradiction.

Scheme of the Proof (continued):

• Let $\iota : K_2 \hookrightarrow C_{2r+1}$ map the edge to the flipped edge of the odd cycle. It induces

$$\iota^*_{K_n} : H^*(\mathtt{Hom}\,(K_2, K_n); \mathbb{Z}) \to H^*(\mathtt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}).$$

• For odd k, using involved spectral sequence computations we can show that $2 \cdot \iota^*_{K_n} = 0$.

• Consider a sequence of $\mathbb{Z}_2$-maps:

$$S_a^{k+1} \to \mathtt{Hom}\,(C_{2r+1}, G) \to \mathtt{Hom}\,(C_{2r+1}, K_{k+3}) \to \mathtt{Hom}\,(K_2, K_{k+3}),$$

note that $\mathtt{Hom}\,(K_2, K_{k+3}) \cong S_a^{k+1}$.

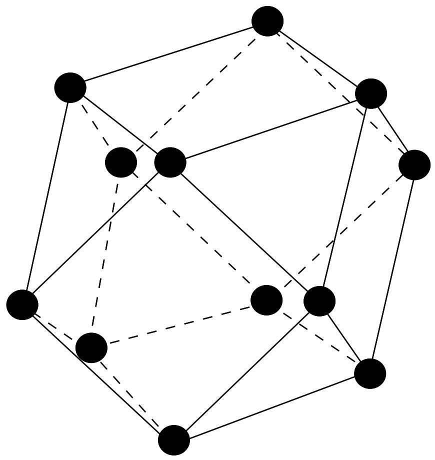
• Combining $h^* := f^* \circ (\phi^{C_{2r+1}})^* \circ (\iota_{K_{k+3}})^* : \mathbb{Z} \to \mathbb{Z}$, we see that h^* is a 0-map.
On the other a $\mathbb{Z}_2$-map $S_a^n \to S_a^n$ cannot induce a 0-map on the nth cohomology group.

Facts about `Hom`'s I.

Theorem. (Babson & K.,2003)

(a) $\mathtt{Hom}\,(K_2, K_n)$ can be realized as a boundary of an $(n-2)$-dimensional polytope, in particular it is homeomorphic to S^{n-2}.

Example.

$\mathtt{Hom}\,(K_2, K_4) =$ 

(b) Let $m, n \in \mathbb{Z}$, $n \geq m \geq 2$. Then $\mathtt{Hom}\,(K_m, K_n)$ is homotopy equivalent to a wedge of $(n-m)$- dimensional spheres.

Facts about $\mathtt{Hom}$'s II.

Theorem. (Babson & K.,2003)

Let T and G be two graphs, and let u, v be vertices of T, such that $N(v) \subseteq N(u)$. Then the inclusion $i : T - v \hookrightarrow T$ induces a homotopy equivalence $i_G : \mathtt{Hom}\,(T, G) \to \mathtt{Hom}\,(T - v, G)$.

Corollary.

(a) Let T be a tree. The map

$$i_{K_n} : \mathtt{Hom}\,(T, K_n) \to \mathtt{Hom}\,(K_2, K_n),$$

induced by an arbitrary inclusion $i : K_2 \hookrightarrow T$, is a homotopy equivalence.

(b) Let F be a forest, then $\mathtt{Hom}\,(\bar{F}, K_n) \simeq \mathtt{Hom}\,(K_m, K_n)$, where M is the maximal cardinality of an independent set in F.

Details of the Proof of the Lovász Conjecture I.

$\mathtt{Hom}_+$-construction.

Similar to $\mathtt{Hom}$, the empty coloring lists are allowed.

- $\mathtt{Hom}_+(T,G)$ is a simplicial complex.

- $\mathtt{Hom}_+(T,K_n)$ is isomorphic to the n-fold join of the independence complex of T.

- There exists a simplicial map

$$\mathrm{supp} : \mathtt{Hom}_+(T,G) \to \Delta_{|V(T)|}\,.$$

- $\mathtt{Hom}\,(T,G) = \mathrm{supp}^{-1}(\rho_0)$, where ρ_0 is the barycenter of $\Delta_{|V(T)|}$.

- The first spectral sequence: the filtration is given by $\mathrm{supp}^{-1}(\Delta^i_{|V(T)|})$, where $\Delta^i_{|V(T)|}$ is the i-skeleton of $\Delta_{|V(T)|}$.

Details of the Proof of the Lovász Conjecture II.

- From the first spectral sequence we conclude that:

  (1) for odd n:

  $$H^{n-3}(\texttt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) = H^{n-2}(\texttt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z},$$

  and $\mathbb{Z}_2$ acts on $H^{n-3}(\texttt{Hom}\,(C_{2r+1}, K_n); \mathbb{Z})$ as a multiplication by -1.

  (2) for even n: $\quad 2 \cdot \iota^*_{K_n}$ is a 0-map.

- The second spectral sequence is used to compute $H^*(\texttt{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$.

- We conclude that

  $$H^{n-2}(\texttt{Hom}\,(K_2, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \to H^{n-2}(\texttt{Hom}\,(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$$

  is a 0-map.

- It follows that $w_1^{n-2}(\texttt{Hom}\,(C_{2r+1}, K_n)) = 0$.

Summary and Outlook

The fact that the Lovász Conjecture is true implies that $\texttt{Hom}\,(-,-)$ construction is interesting and produces nontrivial bounds for the chromatic number.

It is known that the following more general conjecture is false:

$$\chi(G) \geq \chi(T) + \operatorname{conn} \texttt{Hom}\,(T, G) + 1.$$

Is truth somewhere in the middle?

How much information about the graph colorings is contained in the algebro-topological invariants of the $\texttt{Hom}$-complexes?

In the future we would like to connect the topological theory of obstructions with the questions on nonexistence of structural maps in discrete mathematics in a fundamental way.

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