

the topology of reconfiguration

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collaborators: a. abrams [emory]; v. peterson [uiuc], s. lavalley [uiuc]

re:configuration spaces

there are numerous settings in which coordination is
handled via configuration spaces

usually `smooth`

points in 2-d or 3-d

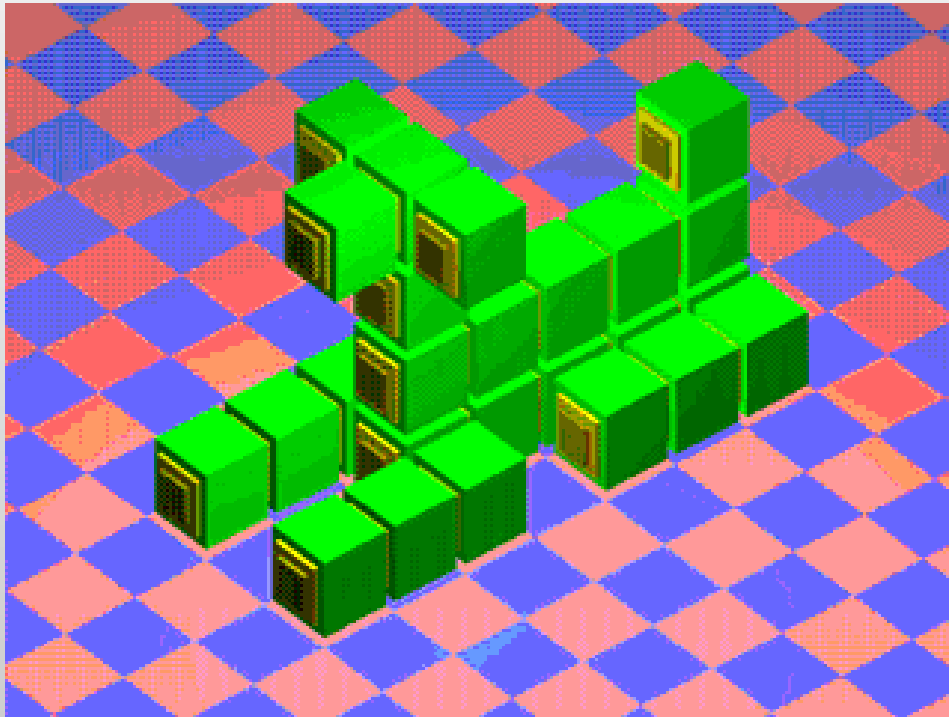
mechanical linkages

vehicle dynamics

however, there are settings for which `discrete` spaces
are more natural...reconfiguration

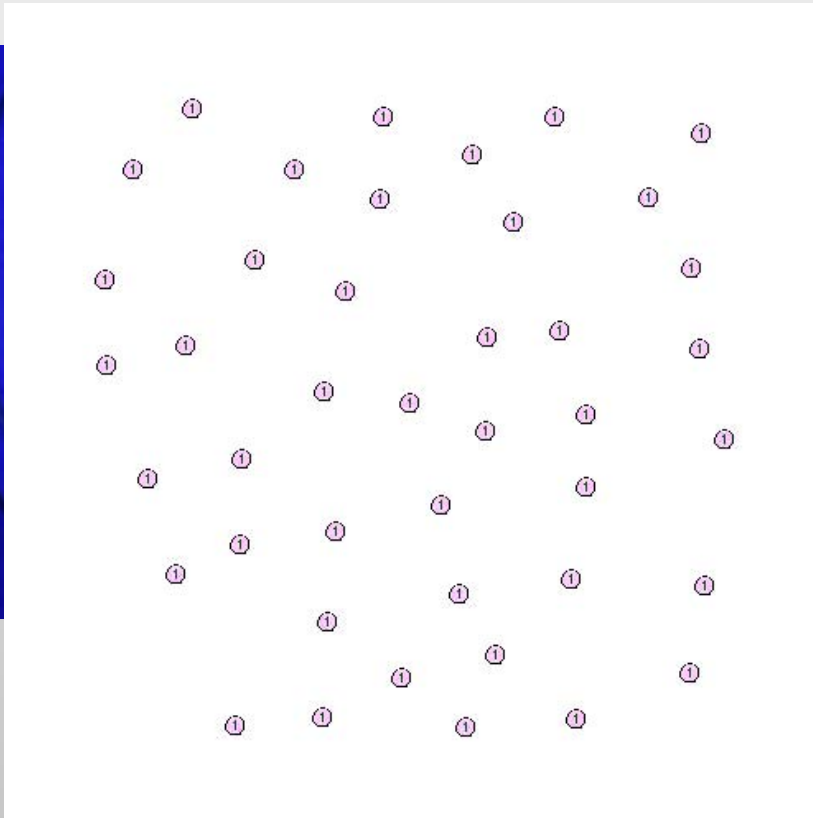
motivation: metamorphic robots

Chirikjian, JHU
Rus & Vona, MIT
Yim, PARC

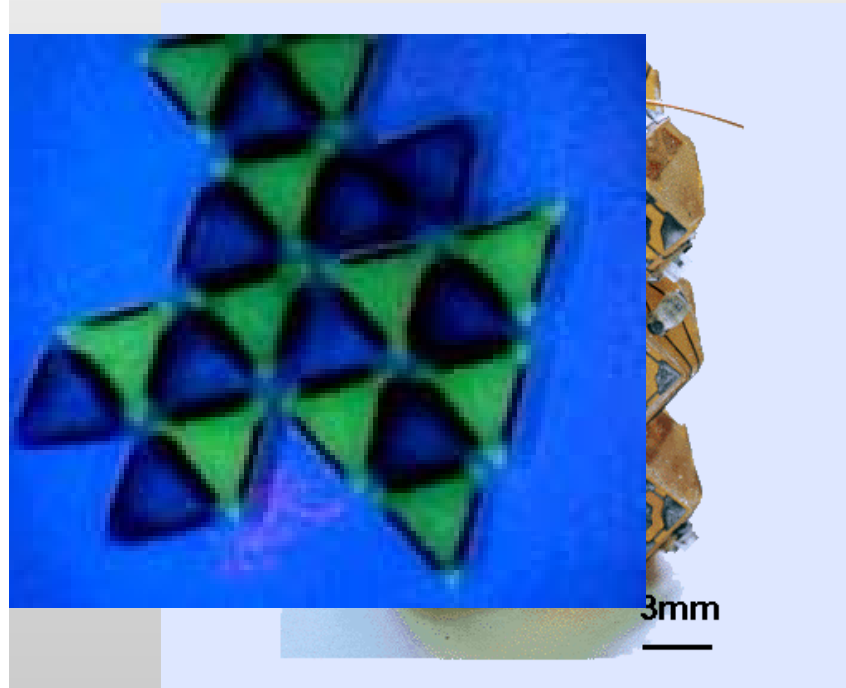


motivation: self-assembly

through passive and active means



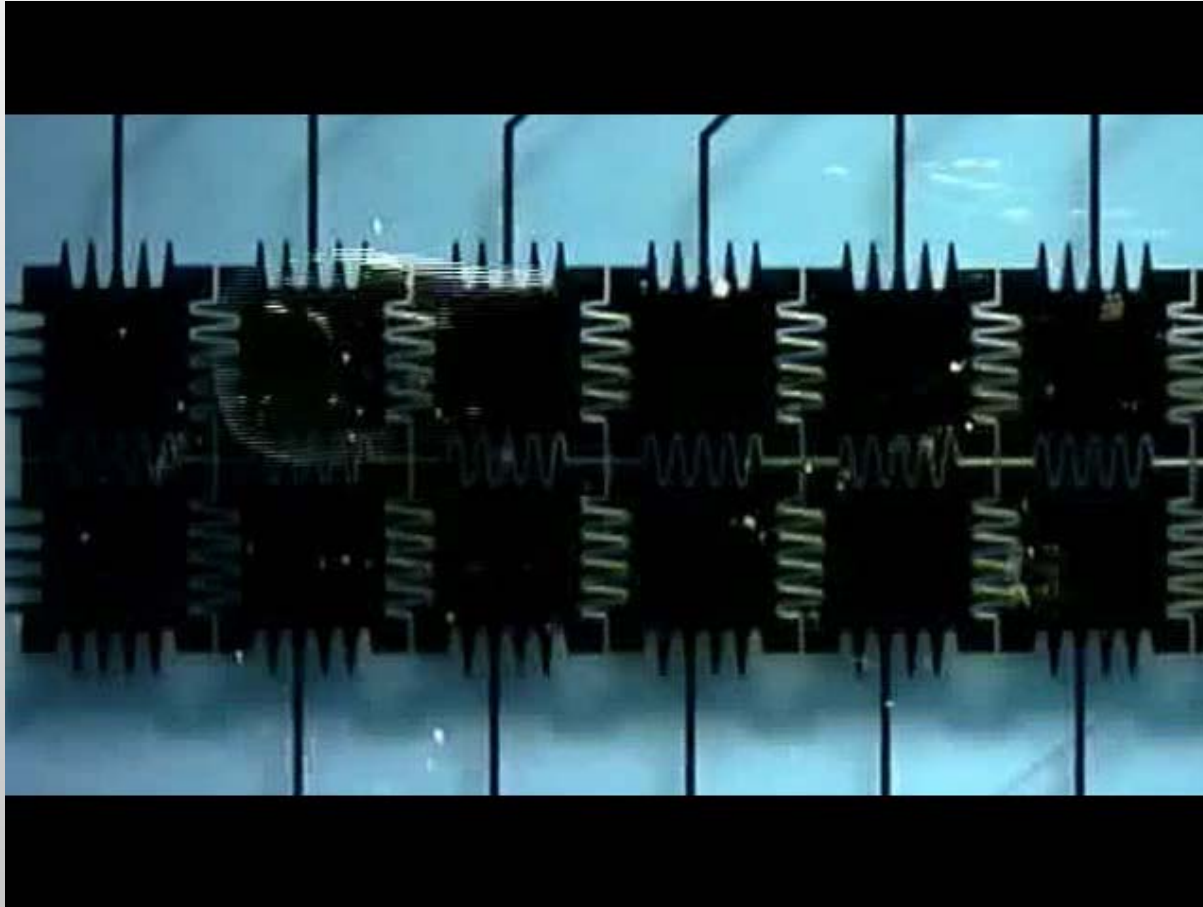
E. Klavins, U. Wash.



G. Whitesides, Harvard

motivation: digital microfluidics

droplet can be manipulated on a grid via electrowetting



duke
univ.

(r. fair)

motivation: group theory

group theory has been using discrete configuration spaces to study groups for well over 100 years...

cayley graphs



more sophisticated geometric objects

this is our perspective...

Definitions and examples...



reconfigurable systems

domain = graph G

(lattice)

states = labelings of vertices of G by an alphabet $\mathcal{A}$

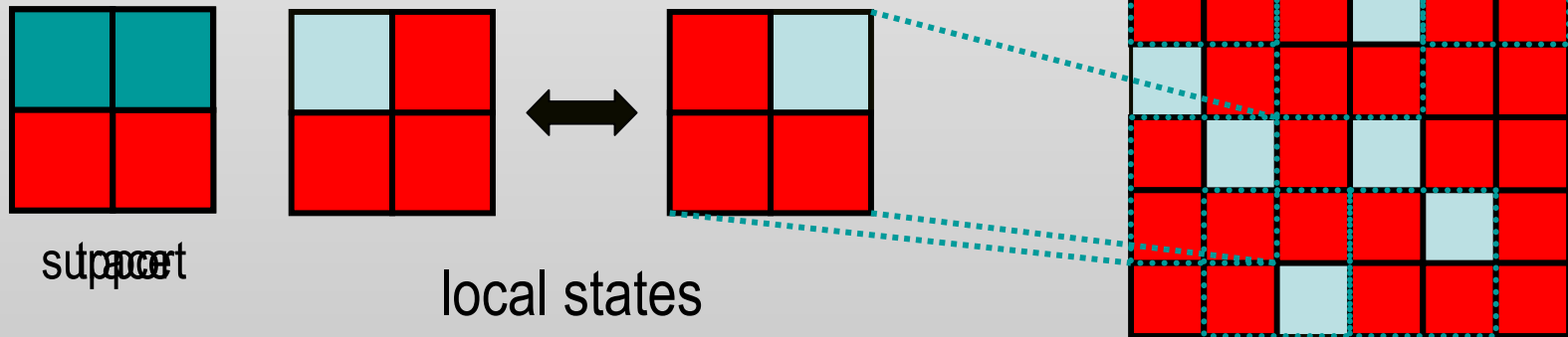
($\mathbb{Z}_2$)

generator = (support, trace; two unordered local states on support)

support = subgraph of G ;

trace = subgraph of support (where things move)

admissible = the support matches a local state

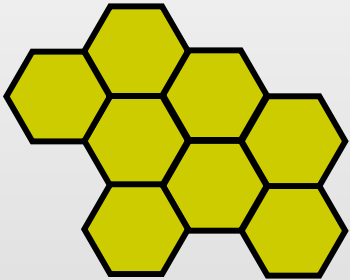


a reconfigurable system is a set of states closed under generators

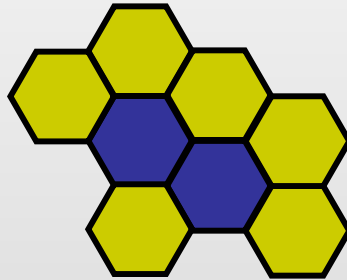
example: 2-d hexagonal

[Chirikjian et al.]

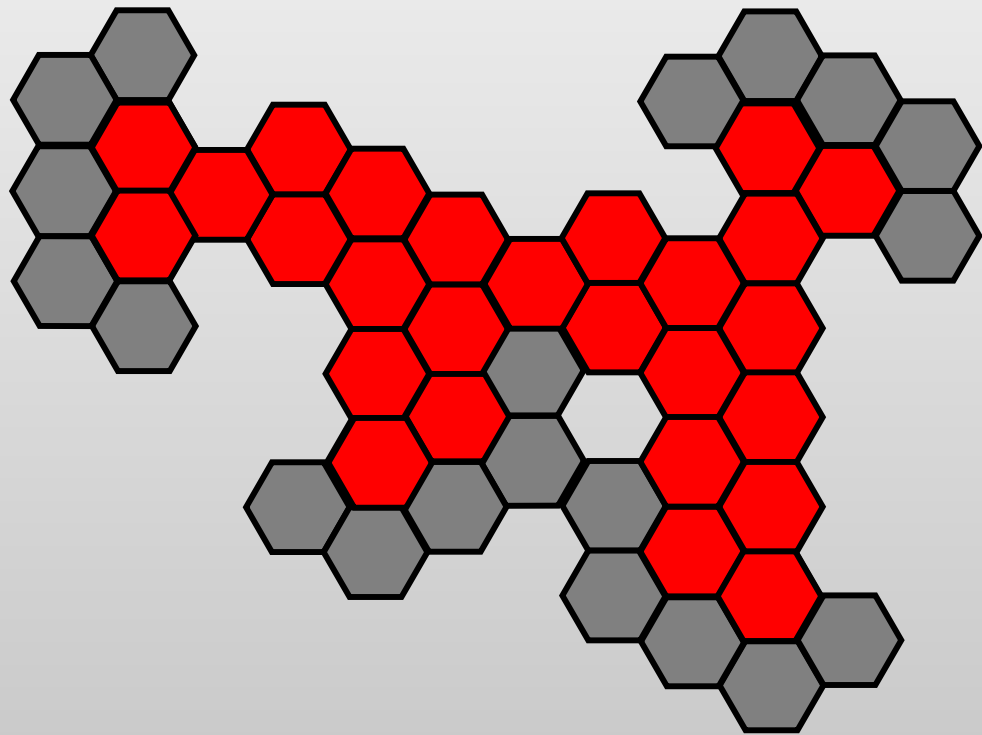
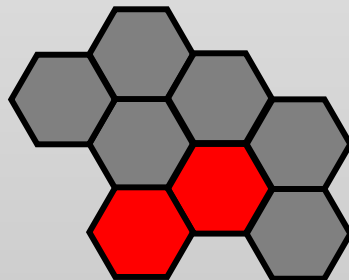
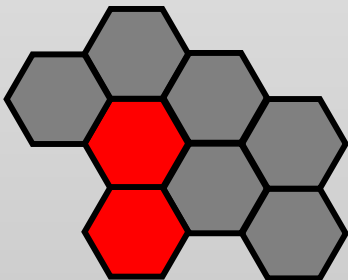
support



trace

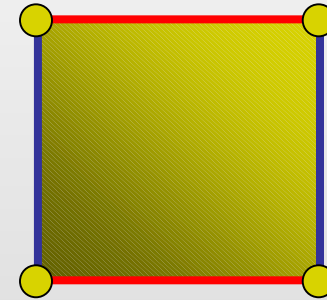
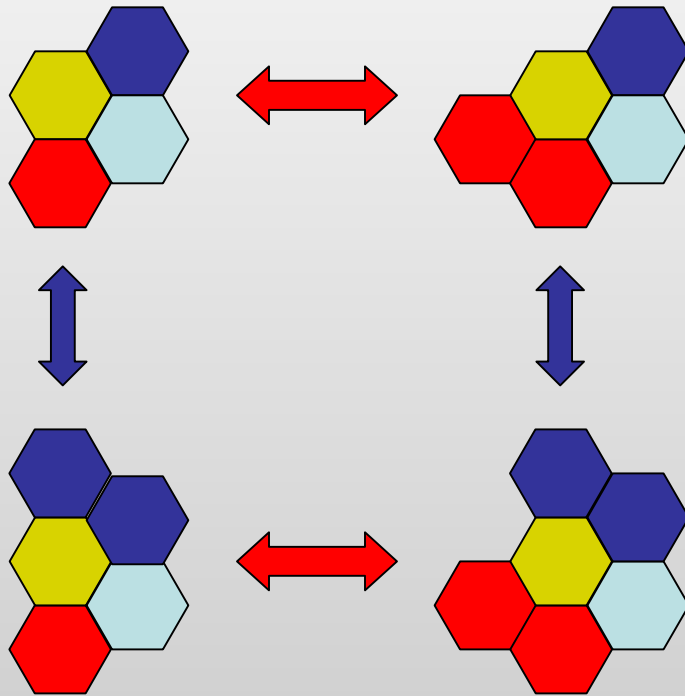


local states



the state complex

idea: transition graph $\Leftrightarrow$ 1-skeleton of a cubical complex



generators $\{ \phi_i \}_{i=1..K}$ commute



$$\text{SUP}(\phi_m) \cap \text{TR}(\phi_n) = 0 : m \neq n$$

each cube of dimension K corresponds to K commutative moves...

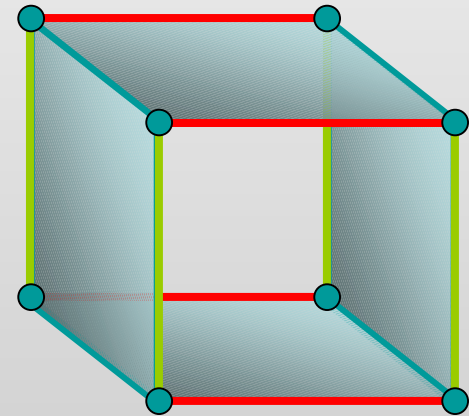
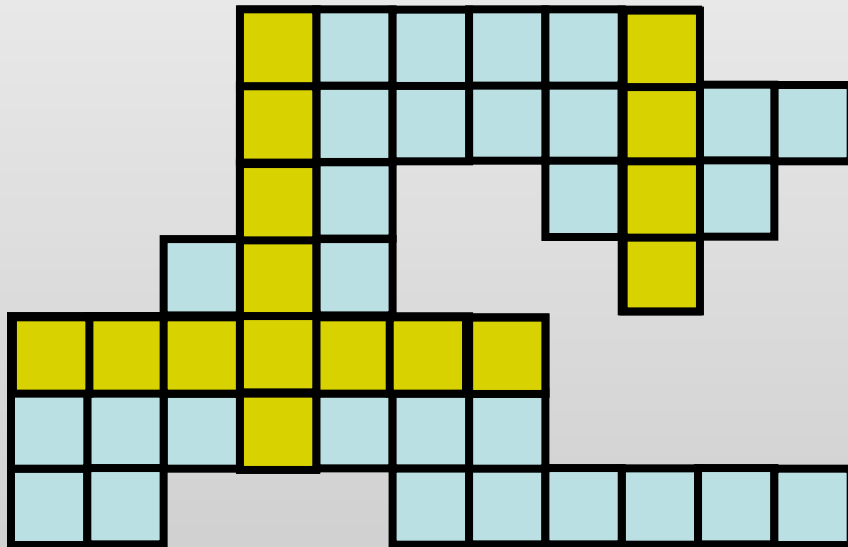
undirected version of **high-dimensional automata** [pratt]

geometric concurrency... [goubault, jardine, raussen, et al.]

example: planar sliding system

tiles: 2-d, square

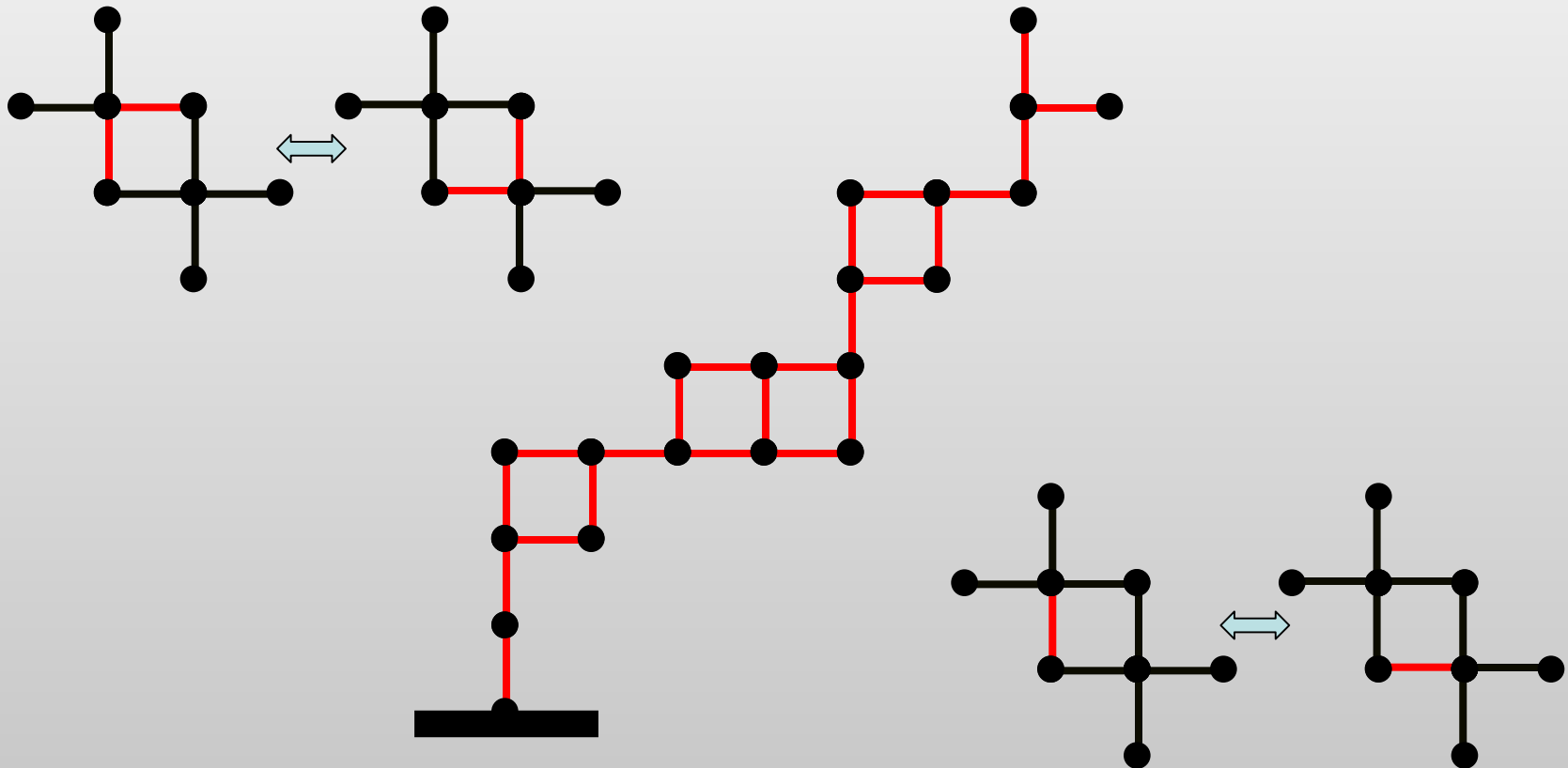
local moves: row, column slides



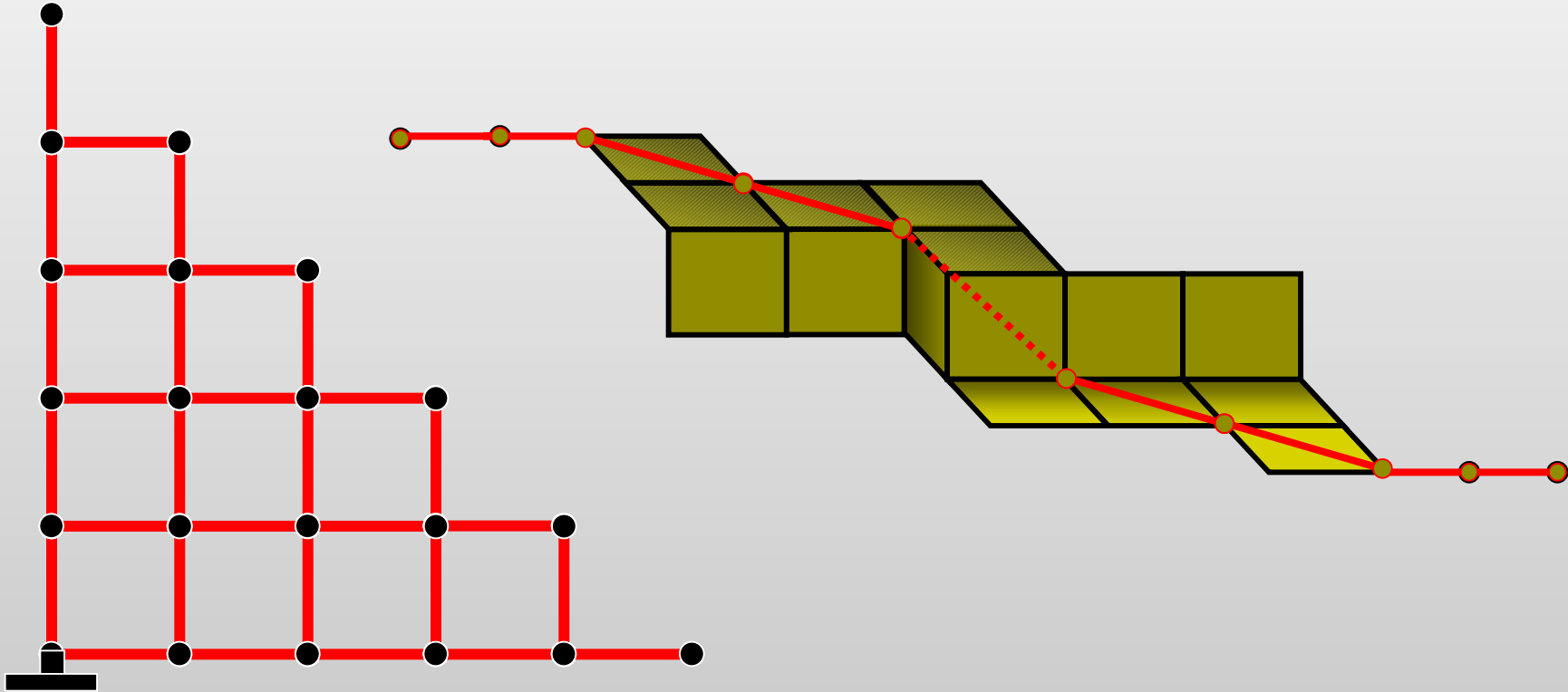
example: articulated robot arm

states = length N chain in planar lattice

moves = rotating end; flipping corners



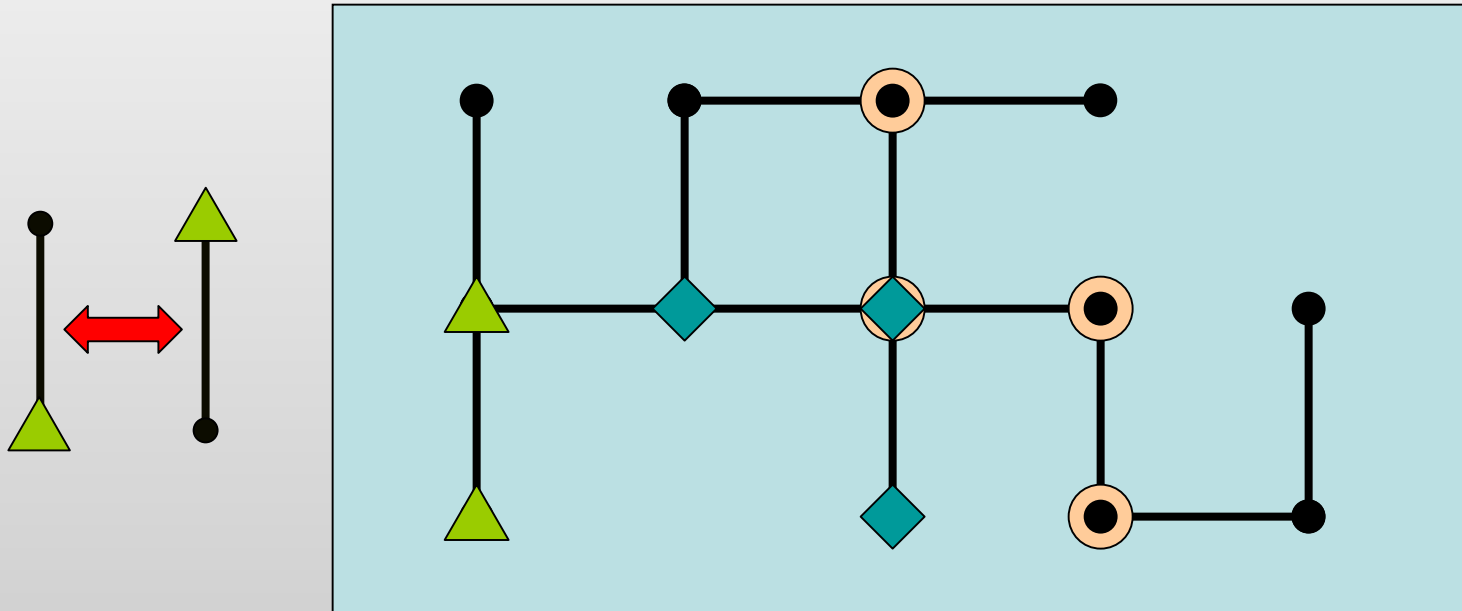
example: articulated robot arm



example: points on a graph Γ

states = arrangements of N points on vertices

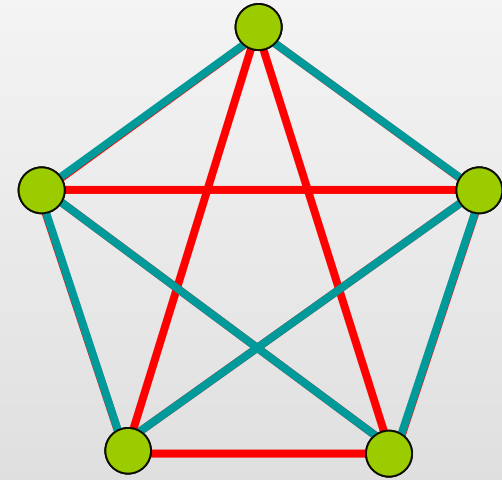
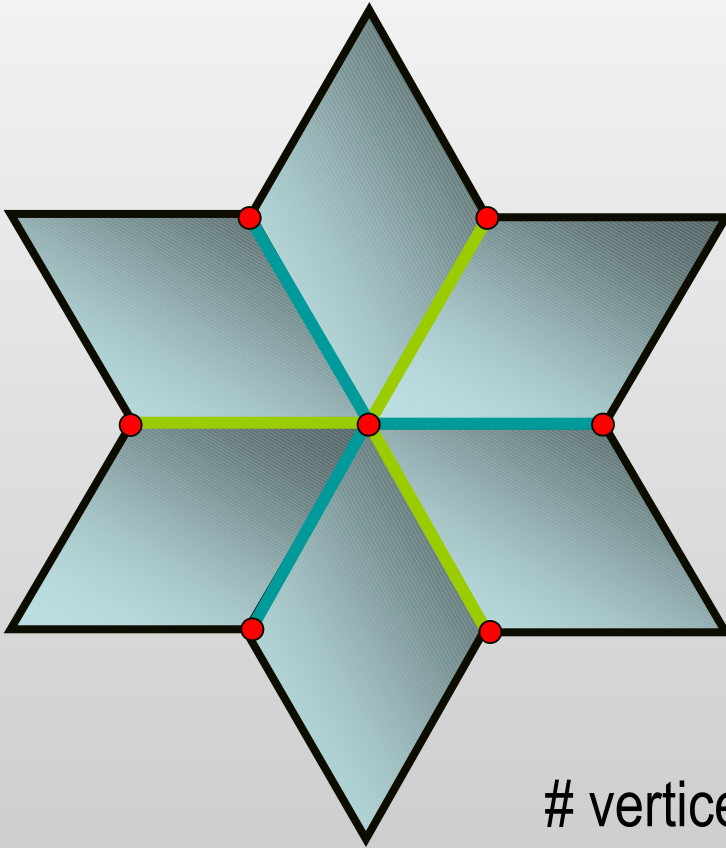
moves = move a point along a free edge



this models microfluidic arrays, as well as configuration spaces
of mobile robots on tracks

configuration spaces of graphs

[A. Abrams, 2000]



1. determine the local structure
2. euler characteristic computation

$$\# \text{ vertices} = (5)(4) = 20$$

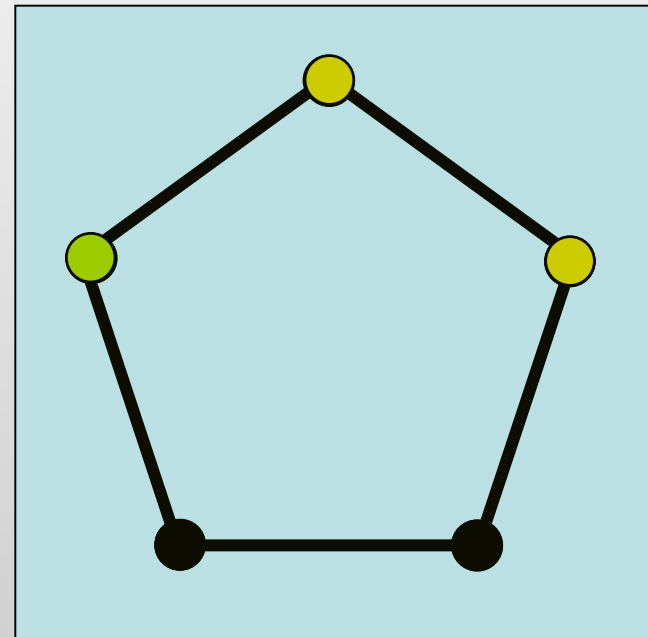
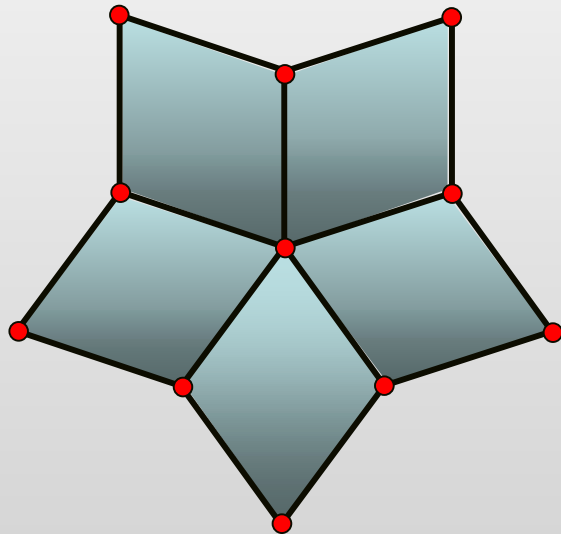
$$\text{local euler characteristic} = 1 - 3 + 3/2 = -1/2$$

$$\text{euler characteristic} = -10 \Leftrightarrow \text{genus} = 6$$

labeled graphs and permutohedra

states = labeled vertices of a graph

generators = exchange distinct labels on neighboring vertices



5-gon with 5 labels: genus 16

filled-in cayley graph of S_5 via transpositions

notice...

all of these surfaces we construct as state complexes are
of genus > 1

can you construct a sphere?

can you construct interesting three-manifolds?

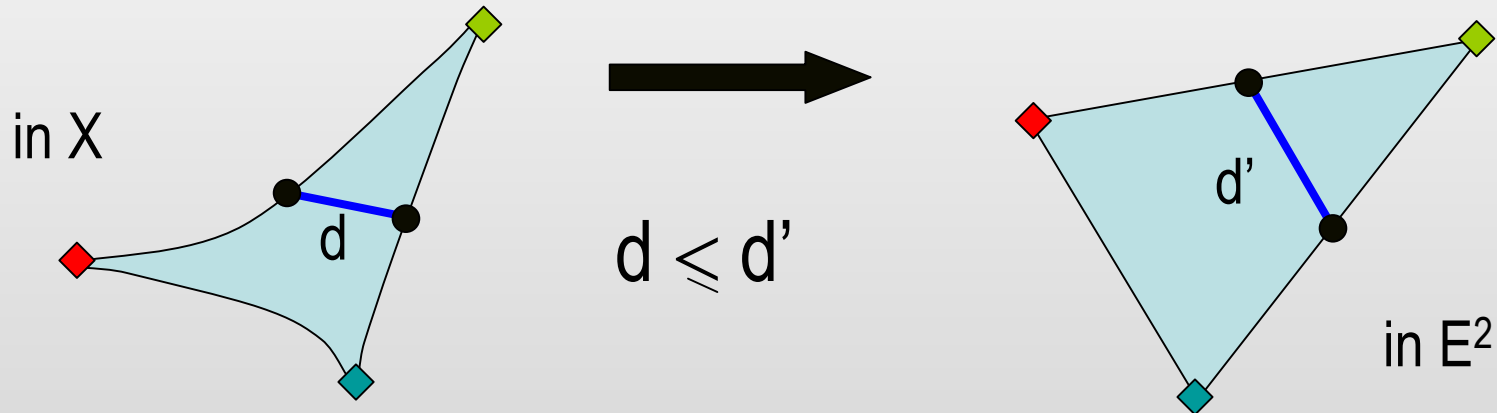
surprisingly, the answer to these **topological** questions
depends almost completely on the **geometry**
of these objects...

and now for some geometry...



cat(0) geometry:

let X be a metric space with local geodesics...



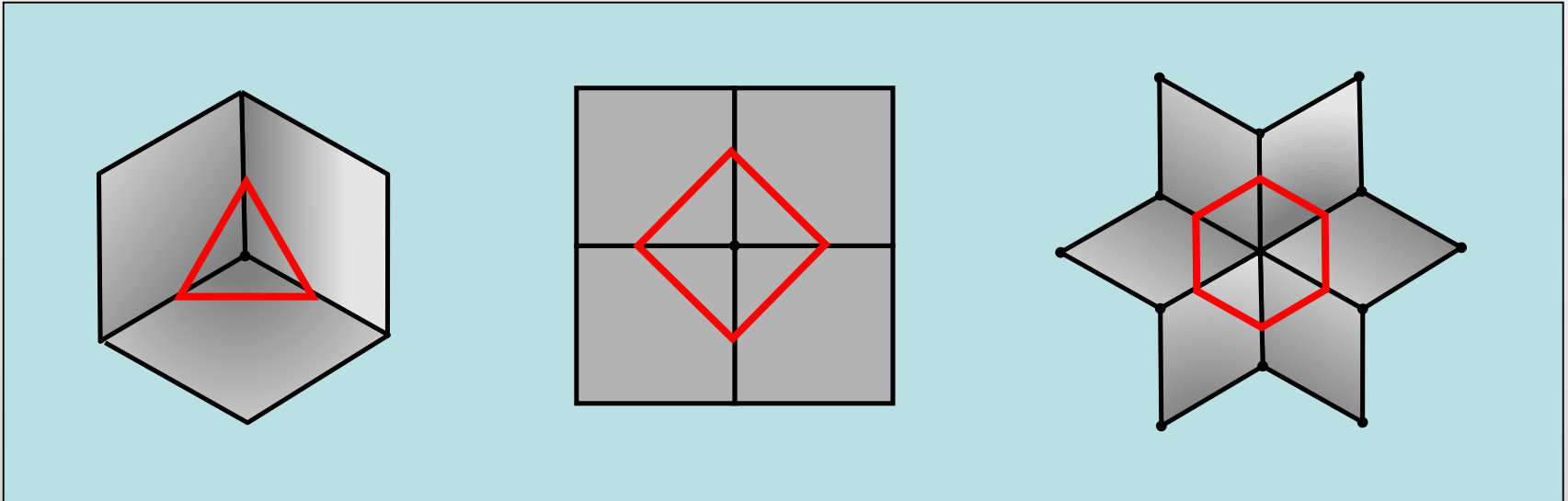
X is **cat(0)** $\Leftrightarrow$ all triangles are no fatter than in E^2

X is **nonpositively curved [npc]** iff X is locally cat(0)
(cat(0) = npc + simply connected)

this approach to geometry has a rich history...

gromov's link condition:

link: simplicial complex of incident cells...



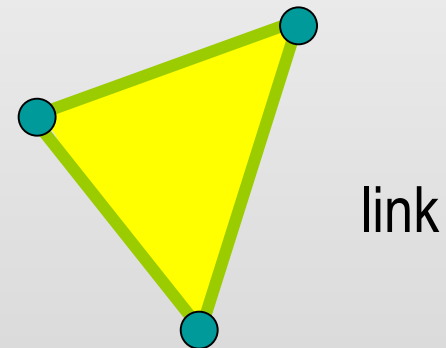
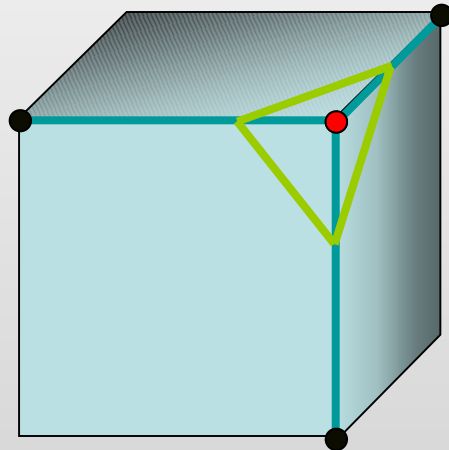
theorem:

cube complex is npc $\Leftrightarrow$ link of each vertex is a **flag complex**

if the edges look like a k -simplex, there
really is a k -simplex spanning them...

gromov's link condition:

link: simplicial complex of incident cells...



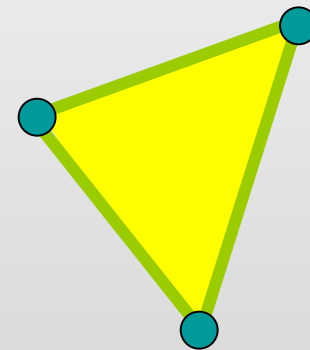
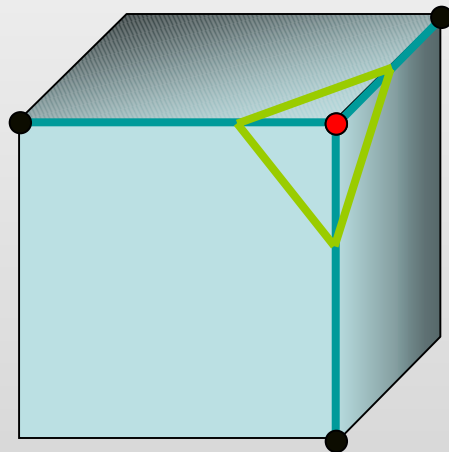
theorem [gromov]:

cube complex is npc $\Leftrightarrow$ link of each vertex is a **flag complex**

if the edges look like a k-simplex, there
really is a k-simplex spanning them...

theorem:

[A+G] all state complexes are nonpositively curved.



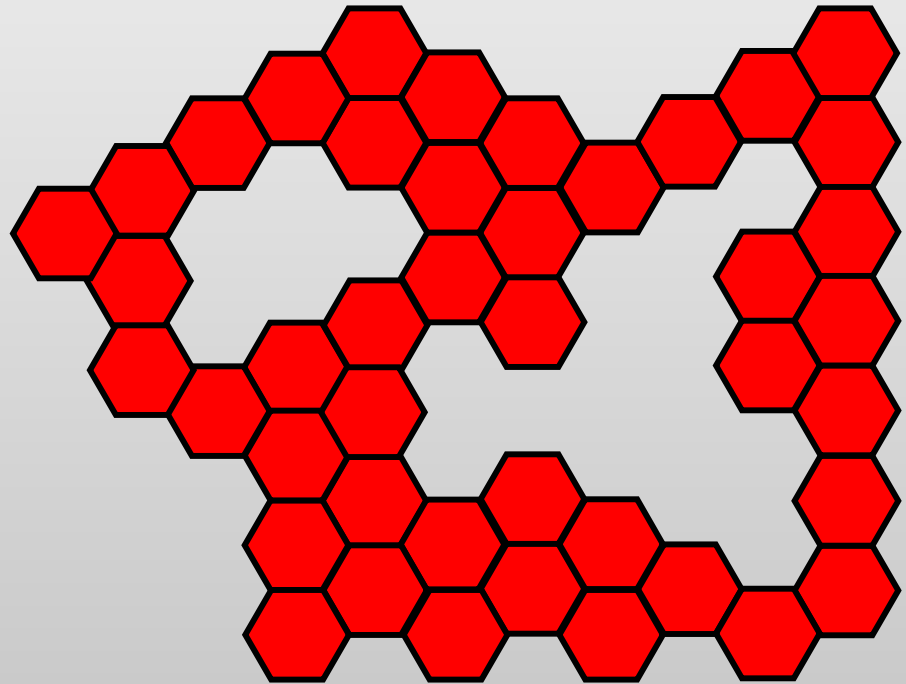
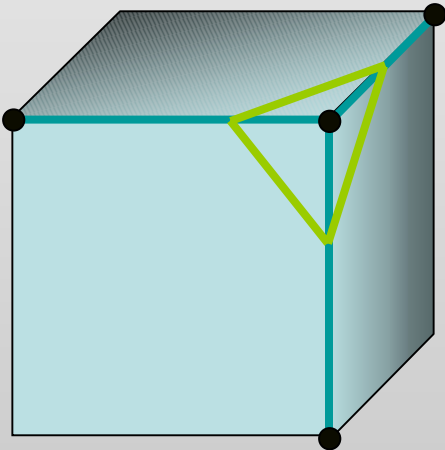
link

proof: simple application of the link condition
commutativity as defined is pairwise determined...

corollary: all higher homotopy groups vanish;
 π_1 is torsion-free

where npc can fail...

note: you really need **local** rules...



the realization problem:

now we know what to look for: what is possible?

proposition: let L be any simplicial complex which is flag. Then there exists a reconfigurable system whose state complex has L as the link of each and every vertex.

proof: $G = L$; one generator for each vertex v in L
support = L -boundary(star(v))
trace = v
change the label on v from 0 $\Leftrightarrow$ 1

show that two 'vertex generators' commute iff they share an edge

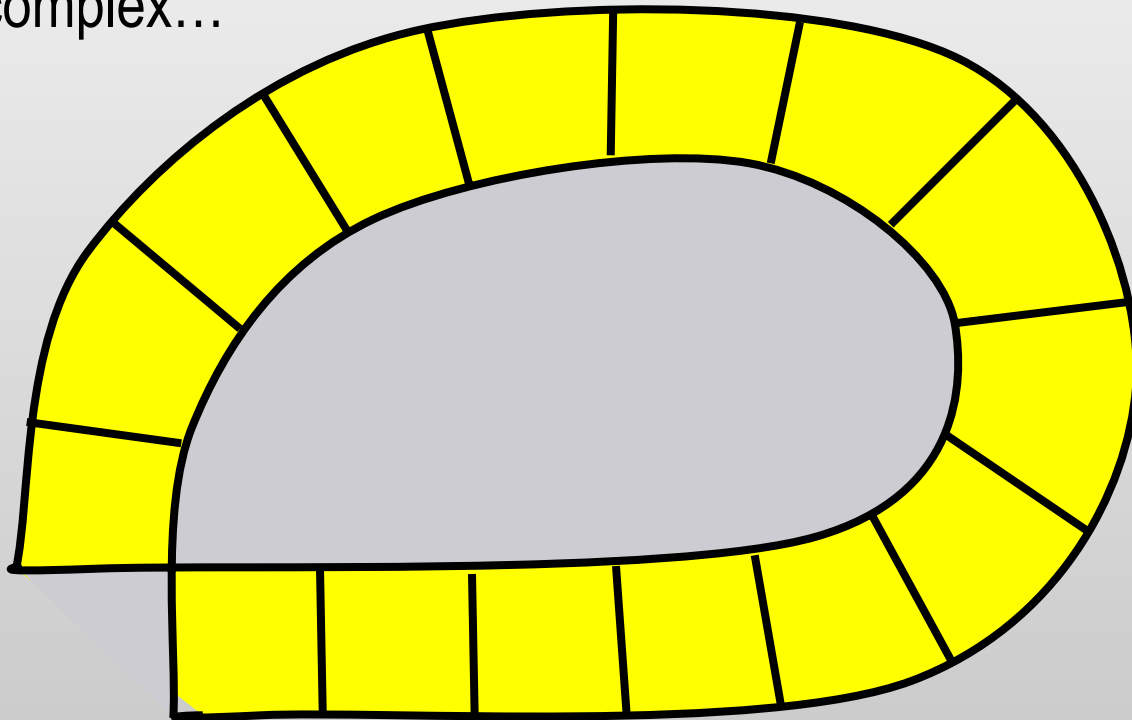
(cf. results of M. Davis in geometric group theory...)

this yields closed 'hyperbolic' manifolds of all dimensions

the realization problem:

so...is anything not possible?

yes: the following npc space cannot arise as a subcomplex of a state complex...



conjecture: this is the only obstruction.

toward a complete characterization...

theorem: [AGP] any npc subcomplex of a product of graphs can be realized as a state complex.

this has a partial converse:

theorem: [AGP] any simply connected state complex is a subcomplex of some cube (of sufficiently high dimension)

the best completion of which would be the following:

conjecture: any state complex can be realized as a subcomplex of a product of graphs...

well ok then...



but what is this good for?

theorem: [AGP] the space of all time-minimal paths between two points in a state complex has the homotopy type of a finite set of points.

proof:

fact: geodesics are unique on a $\text{cat}(0)$ space

fact: time-minimal paths $\Leftrightarrow L^\infty$ geodesics

deform the set of L^∞ geodesics to the unique geodesic

for an npc space, do this in each homotopy class

compactness argument: finite number of classes

but what is this good for?

this is very useful for optimization problems

stratagem:

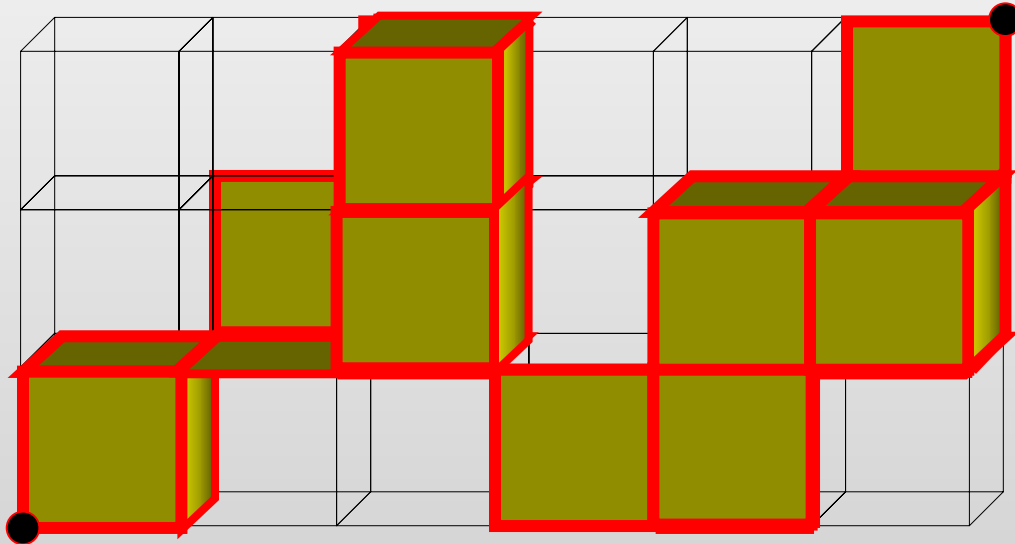
given some state trajectory from a distributed, ad hoc,
or probabilistic path-planner...

1. perform curve shortening on state complex
2. this **must** converge to the **global** minimum
obtainable from the initial path...

npc => no local minima at which to get stuck...

curve-shortening algorithm

given: edge-path in S
march along checking commutativity
multiple sweeps through path reduce length...



theorem [AG]: converges to the global minimal-time state trajectory obtainable from initial path.

you do not need to construct the entire state complex

optimal scheduling

let $\Gamma = (\Gamma_1, \dots, \Gamma_N)$ denote N graphs, denoting either :
roadmaps for N different robots in a C -space;
 N different processes with shared resources

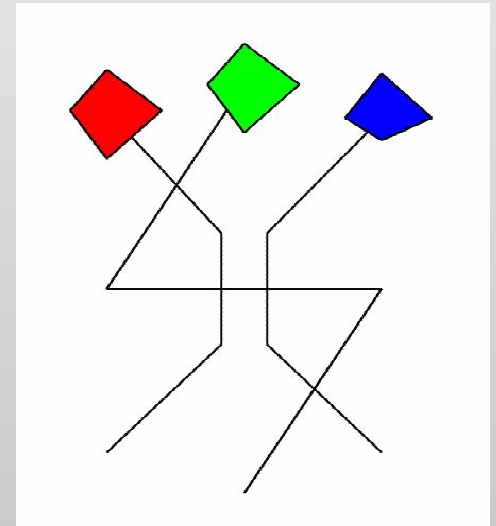
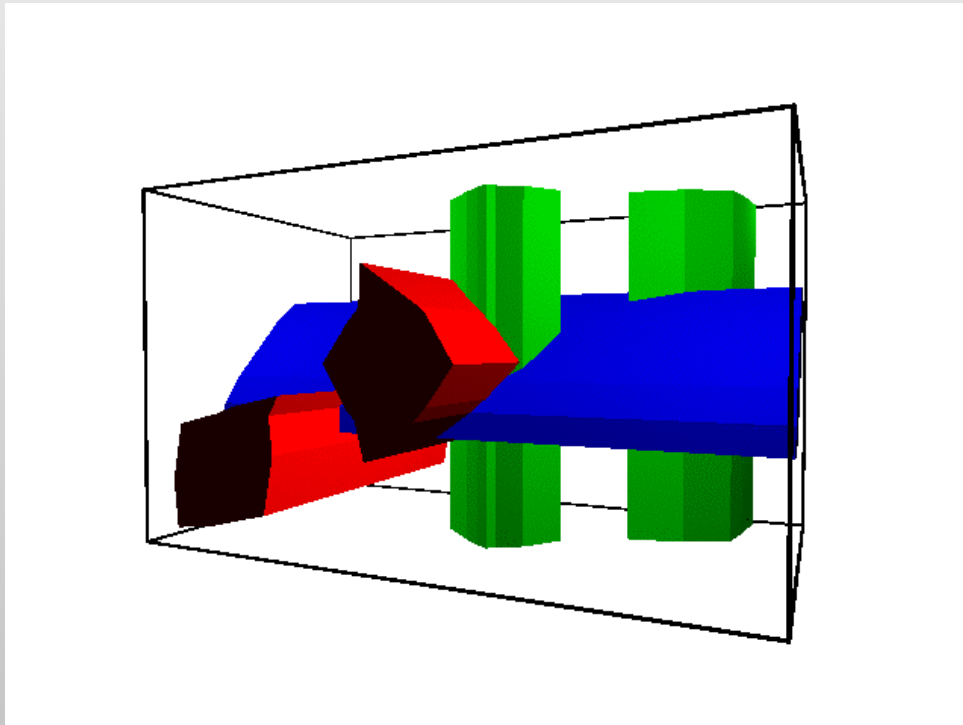
coordination space:

$$C(\Gamma) = \Gamma_1 \times \dots \times \Gamma_N - O$$

O = obstacle set (open) where, e.g.,
robots collide or processes interfere...

optimal scheduling

for obstacle sets O defined by collisions,
the coordination space is **cylindrical**...



pareto optimality

goal: optimal coordination/scheduling

but: each robot/process has its own cost
function (e.g., elapsed time)

we –could- use

average cost

maximal cost

nonlinear weighted cost?

use pareto-optimization



pareto optimality

definition: a path is **pareto-optimal** iff it is minimal with respect to the partial order on cost vectors.

cost vectors $A=(a_1,\dots,a_N)$; $B=(b_1,\dots,b_N)$

$$A \leq B \Leftrightarrow a_i \leq b_i \text{ for all } i$$

incomparable if

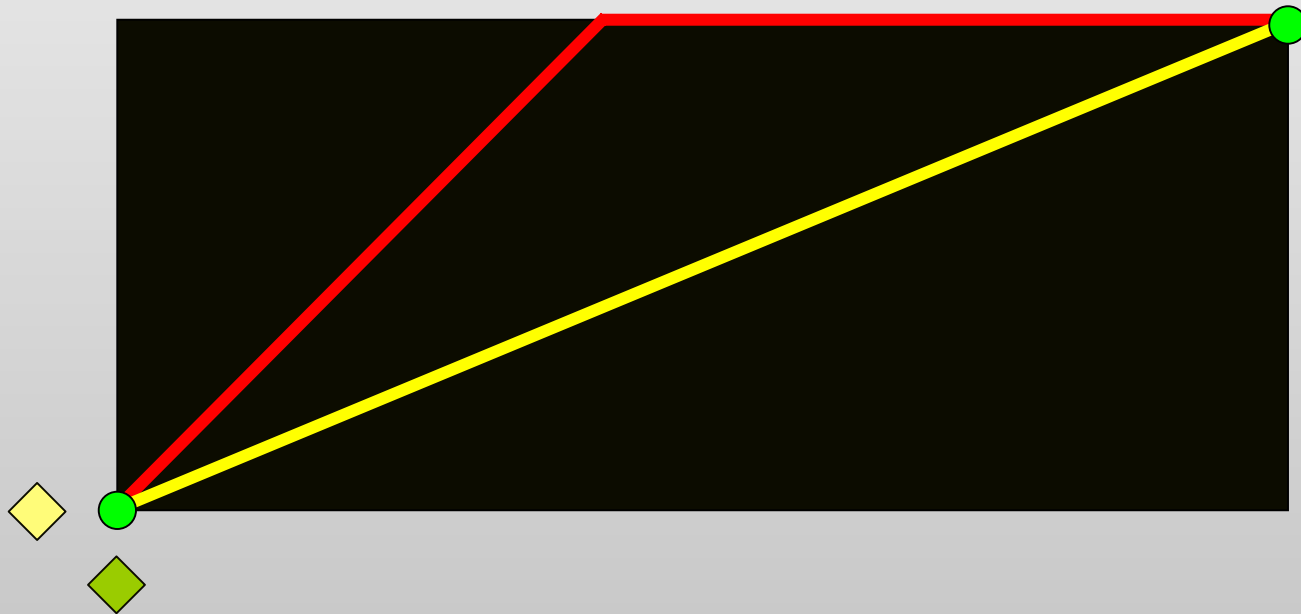
$$a_i < b_i \text{ and } a_j > b_j \text{ for some } i, j$$

equivalent if

$$a_i = b_i \text{ for all } i$$

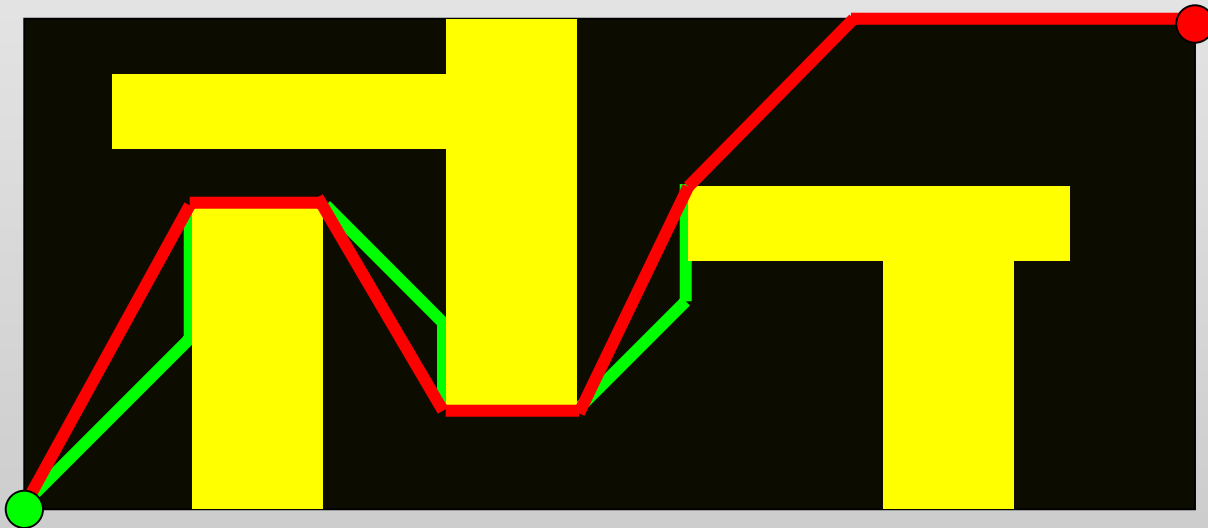
pareto optimality

problem: classify & compute pareto optimal path classes



classes

classes



why pareto?

lemma: any optimum for any monotone scalarization of the cost functions is in fact a pareto optimum.

precomputing the pareto optima is good:

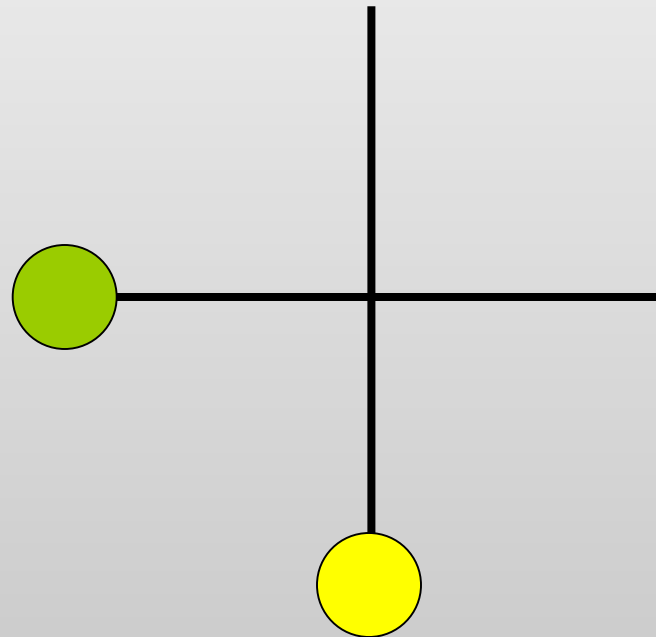
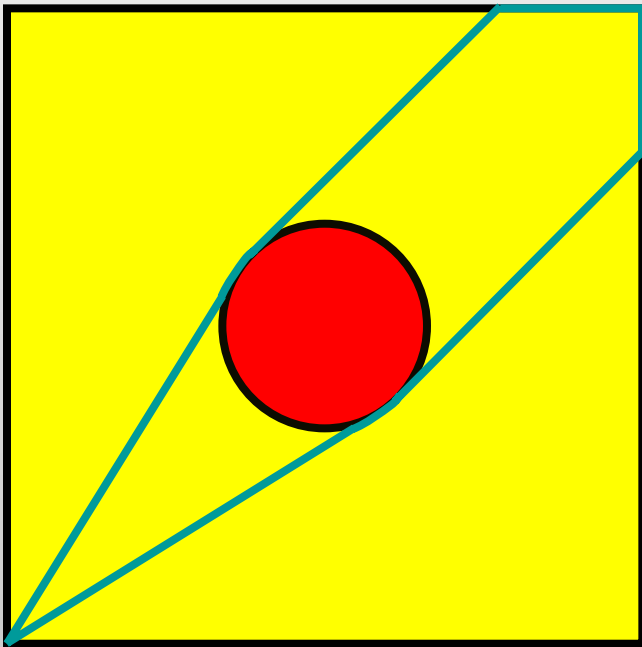
set of all possibilities

changing priorities / costs

hopefully, this is a small set of optima

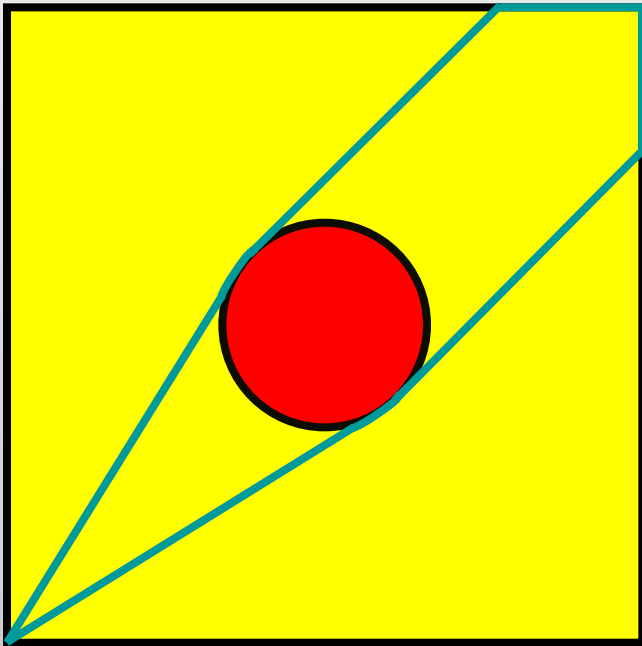
example

cylindrical

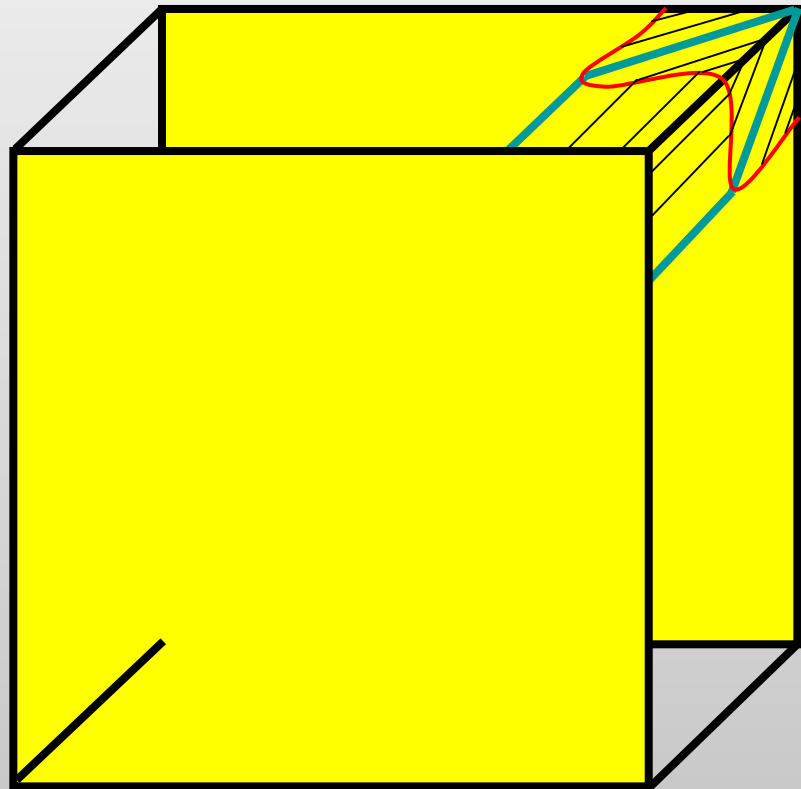


example

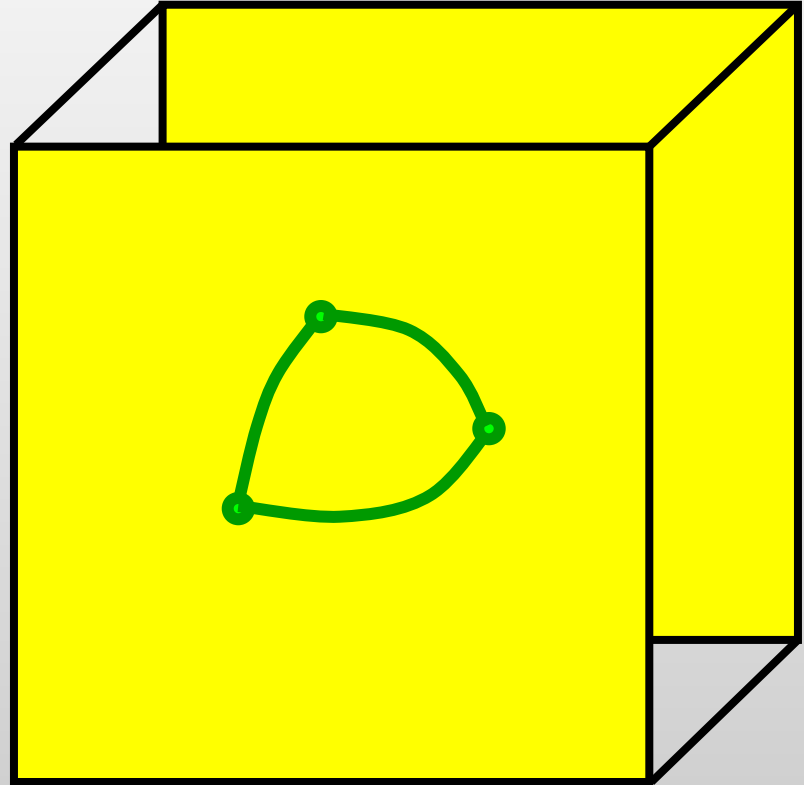
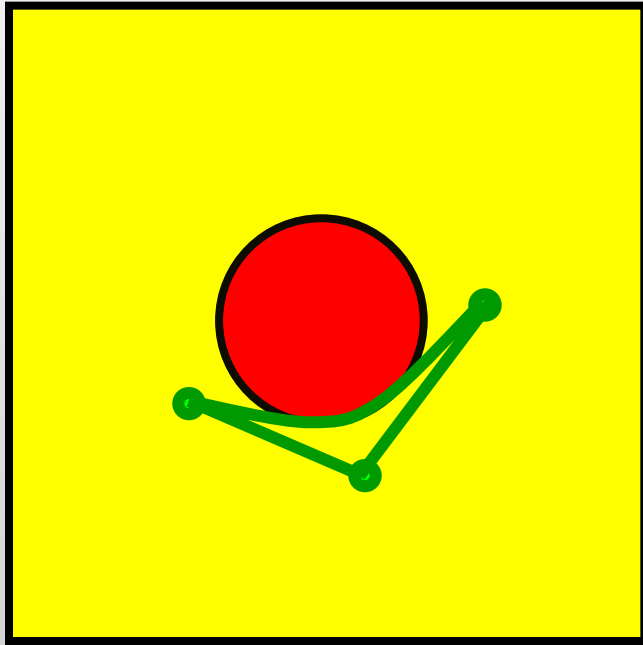
cylindrical



noncylindrical



what is the difference?



(non)positive curvature

pareto optimality

theorem: [GL] for C cylindrical (obstacles defined by pairwise collisions), there is a finite number of pareto optimal classes

proof:

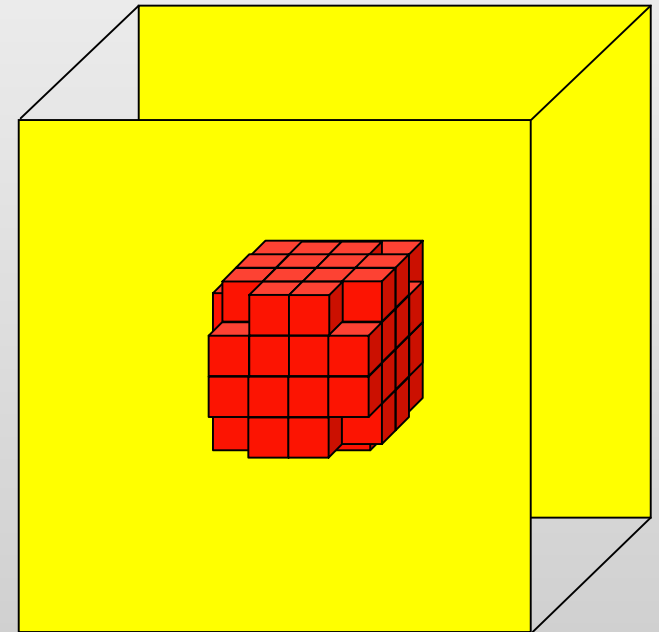
- * given a pareto-optimal path in C
- * approximate it by a cube path in $C^{(n)}$
- * deform it to a **left-greedy cube path** (unique on npc space)
- * verify that cost functions have not changed
- * use compactness to go from discrete to finite

pareto optimality

why doesn't this work in the case
of positive curvature?

...all the steps work
except one

- * can approximate by cube paths
- * left-greedy cube paths are
pareto optima
- * there is no longer a unique left-greedy path



exponential blow-up in #paths

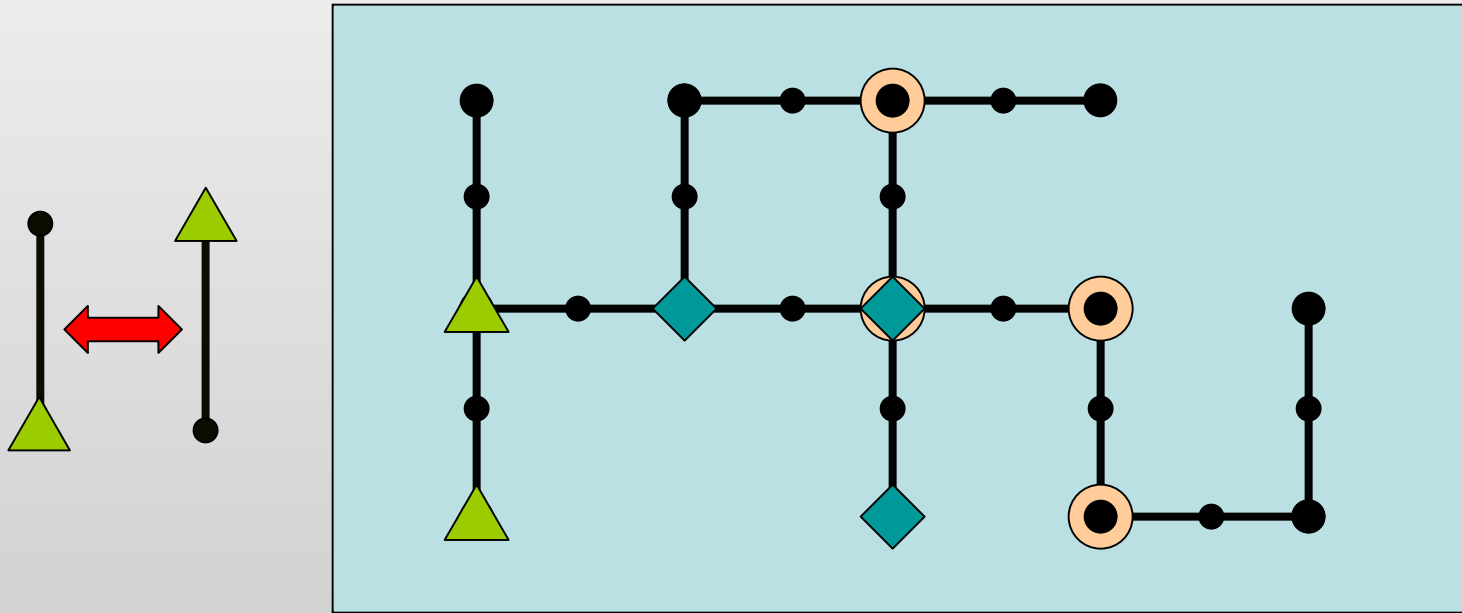
open questions...



open question 1:

convergence

what is the 'limit' of a reconfigurable system under refinement?



theorem: [a. abrams] refinements converge in homotopy type to the ('smooth') configuration space of the graph

open question 1:

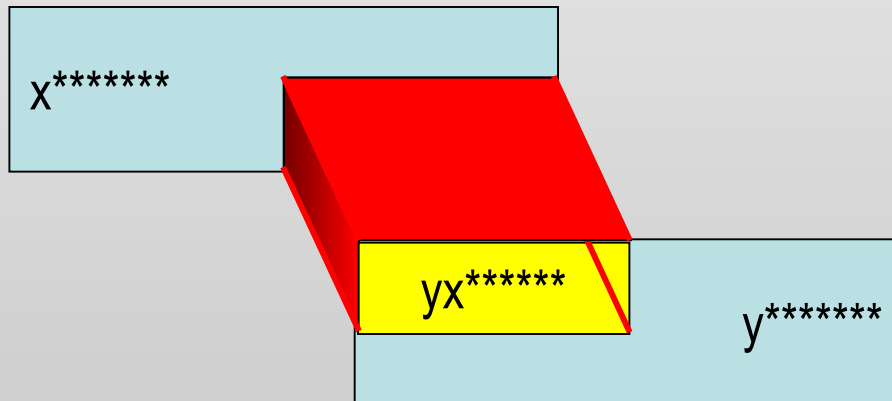
convergence

proposition: refinements of the articulated arm always give contractible state complexes

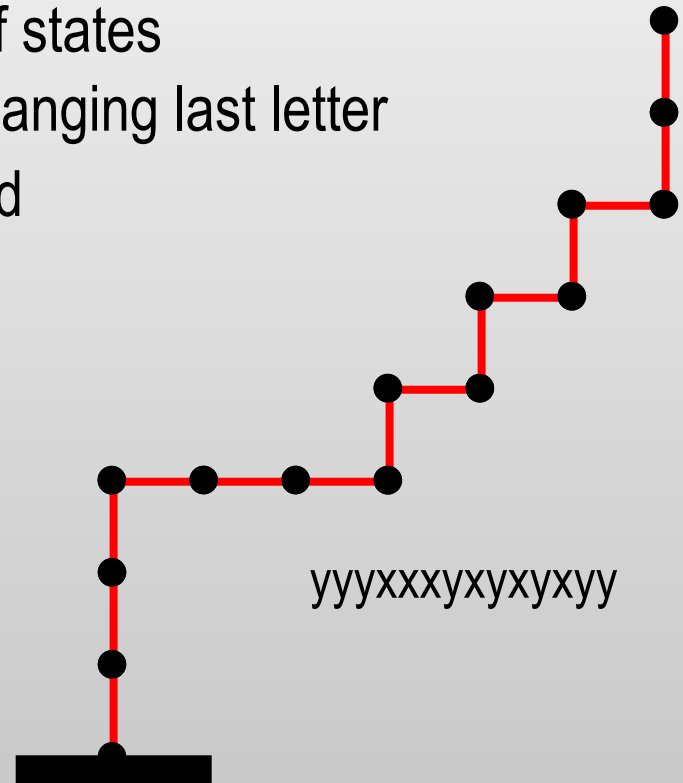
proof: use symbolic representation of states

actions = transposing $x \Leftrightarrow y$; changing last letter

induct on the length of the word



state complex “converges” to a smooth configuration space



open question 1:

convergence

when do state complexes converge to a smooth configuration space under refinement?



open question 2:

complexity

meta-theorem: algorithmic problems are

intractable

quadratic

linear

for groups of

positive

nonpositive

negative

curvature

cf: shortest path problem in 3-d

np hard [canny-reif]; but this relies on positive curvature...

recall: on a $\text{cat}(0)$ space, there is a unique geodesic



open question 2:

complexity

is it true that $NP + CAT(0)$ yields P ?

