Homotopy types of box complexes

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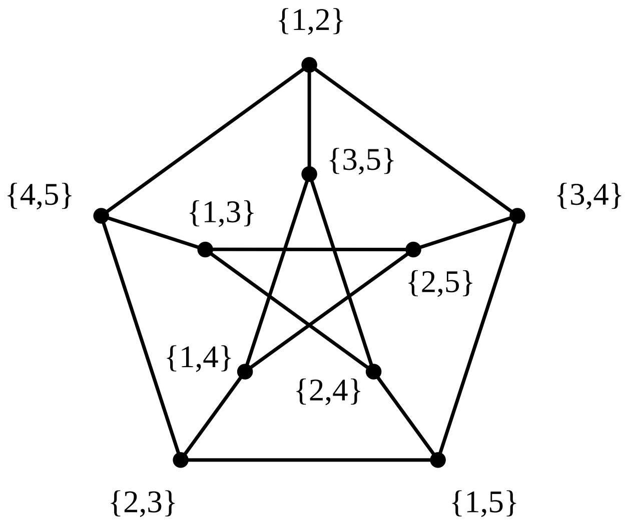
Combinatorics
Geometry
Computation

Kneser's Conjecture (1955)

Kneser graph $KG_{n,k}$:

1. nodes: n-subsets of $\{1, \ldots, 2n + k\}$
2. edges: between disjoint sets



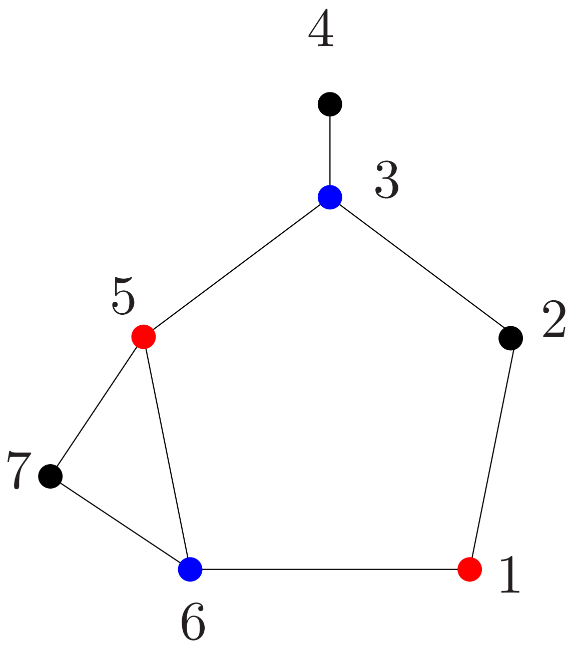
The Kneser graph $KG_{2,1}$ is the Petersen graph:

Theorem (Lovász (1978)). *The chromatic number of the Kneser graph $KG_{n,k}$ is $k + 2$.*

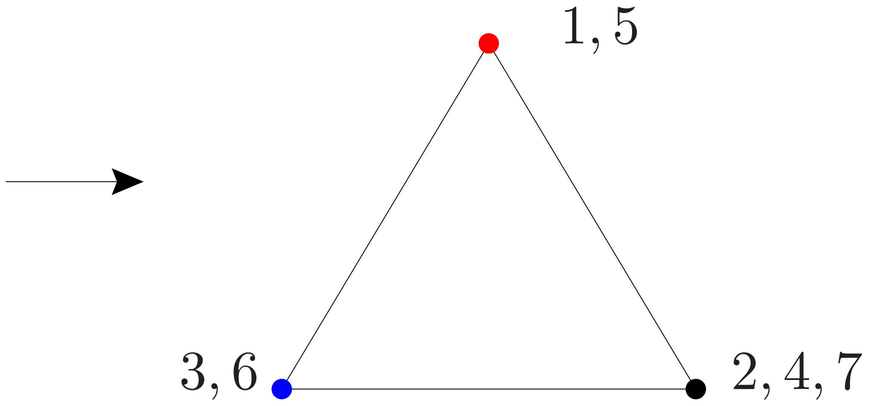
Matoušek (2000): combinatorial proof of Kneser's conjecture.

Chromatic Number

$$\chi(G) = \min\{n \colon \exists G \to K_n\}$$

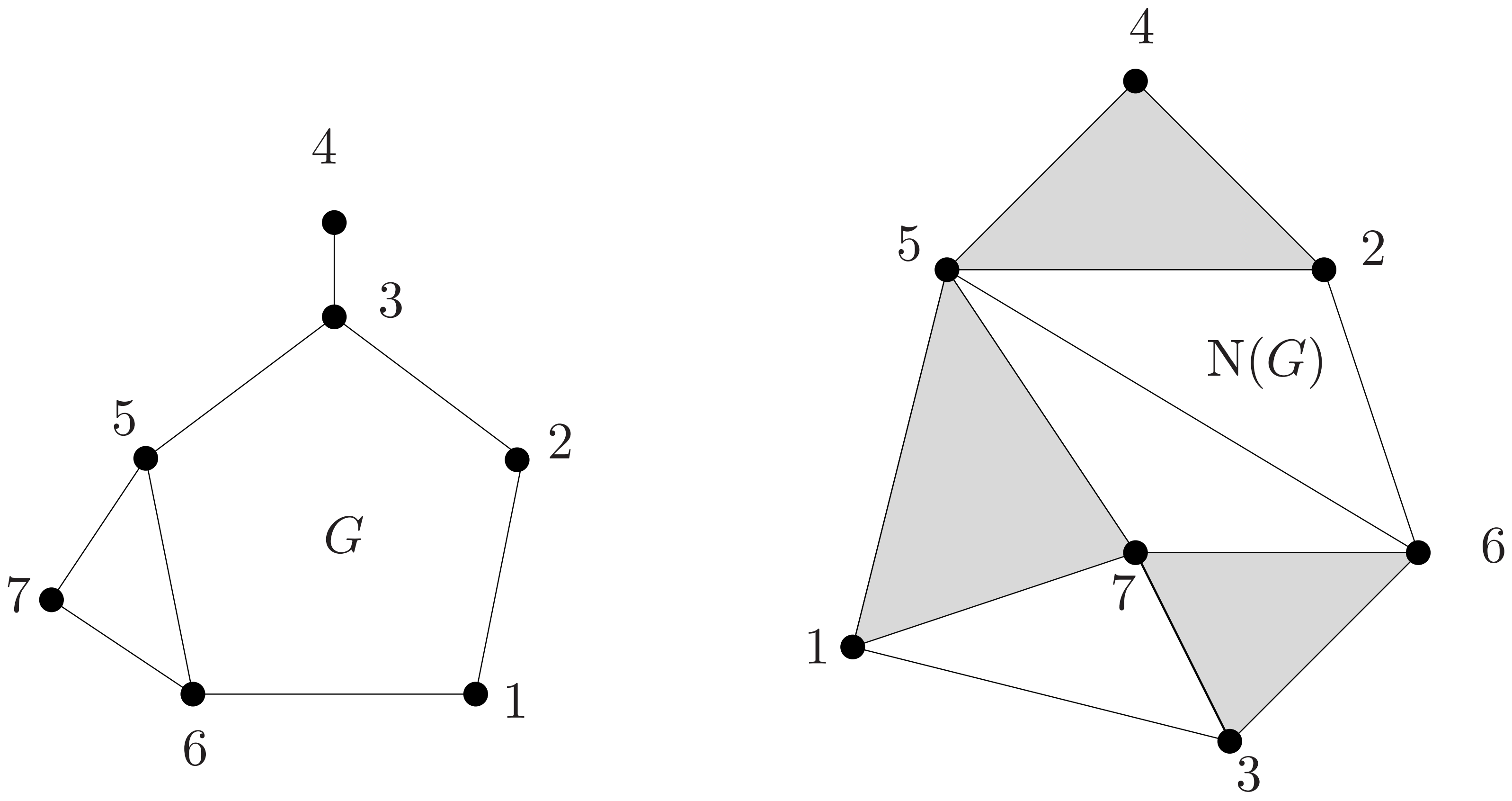


G



Neighborhood Complex

$$\mathrm{N}(G) = \{S \subseteq V(G) \colon \mathrm{CN}(S) \neq \emptyset\}$$



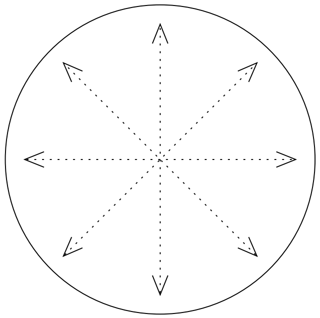
Theorem (Lovász '78). $\chi(G) \geq \text{connectivity}\,(\mathrm{N}(G)) + 3$

Topological Interlude

- $\mathbb{Z}_2$-space: pair (X, ν) such that

$$\begin{tikzcd} X \arrow[r, "\nu"] \arrow[rr, bend right, "\mathrm{id}_X"'] & X \arrow[r, "\nu"] & X \end{tikzcd}$$

- X free: if ν has no fixed point

$(S^1, -)$ 

- $\mathbb{Z}_2$-map $f\colon (X, \nu) \to (Y, \omega)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\nu"'] & Y \arrow[d, "\omega"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

- $\mathbb{Z}_2$-index:

$$\operatorname{ind}(X) = \min\left\{n \geq 0\colon\ X \xrightarrow{\mathbb{Z}_2} S^n\right\}$$

Box Complex $(\mathrm{B}(G))$

$$V(G) = \{v_1, v_2, \ldots, v_n\}$$

$$V(\mathrm{B}(G)) := V(G) \bigcup V(G) = \left\{ \begin{array}{c} v_1, v_2, \ldots, v_n, \\ v_1, v_2, \ldots, v_n. \end{array} \right\}$$

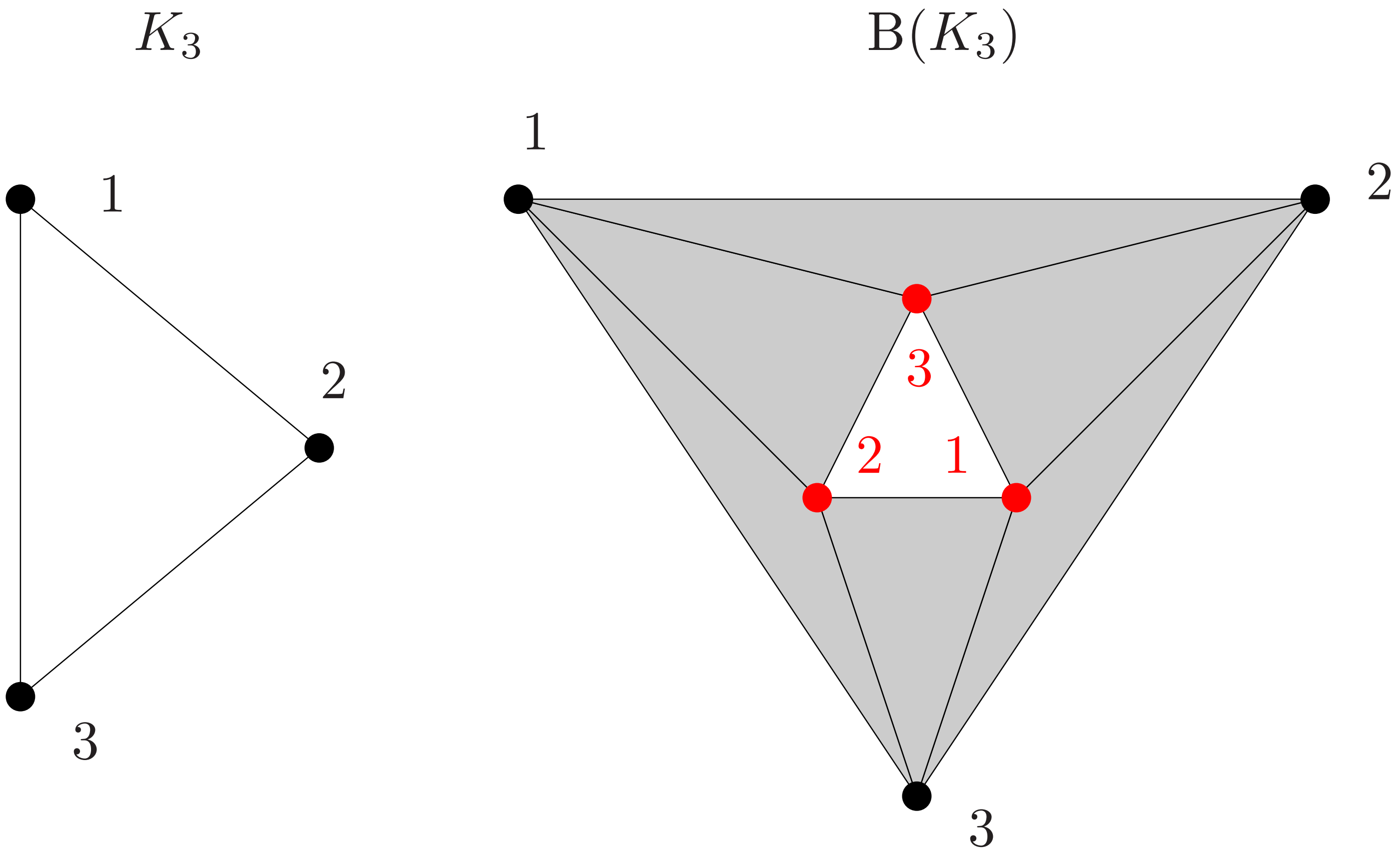
$$\mathrm{B}(G) = \left\{ \begin{array}{ccl} A \cup B & : & A, B \subseteq V(G), A \cap B = \emptyset \\ & & G[A, B] \text{ is complete, } \mathrm{CN}(A) \neq \emptyset \neq \mathrm{CN}(B) \end{array} \right\}$$

Common Neighbors

$$A \subseteq V(G)$$

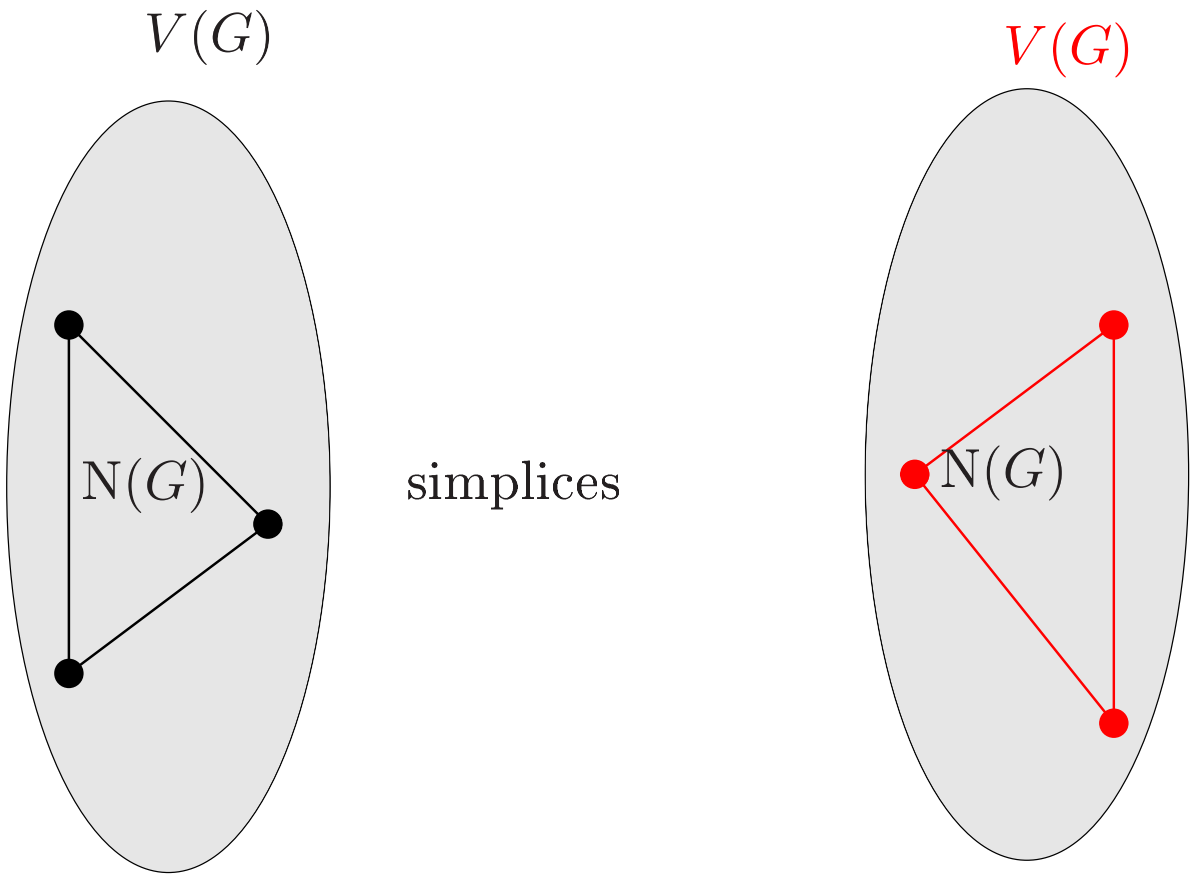
$$\mathrm{CN}(A) = \{v \in V(G) \colon \{a, v\} \in E(G) \text{ for all } a \in A\}$$

Example



$$B(K_n) = S^{n-2} \times [0,1] \quad \text{and} \quad \operatorname{ind}(\mathrm{B}(K_n)) = n-2$$

Box Complex



Advantage

If $f\colon G \to H$ is a graph homomorphism, then

$$v \to f(v) \text{ and } v \to f(v)$$

$v \in V(G)$ defines a $\mathbb{Z}_2$-map $\mathrm{B}(G) \xrightarrow{\mathbb{Z}_2} \mathrm{B}(H)$

Theorem (Lovász). $\chi(G) \geq \operatorname{ind}(\mathrm{B}(G)) + 2$

Proof. $n := \chi(G)$.

$G \to K_n$

$\mathrm{B}(G) \xrightarrow{\mathbb{Z}_2} \mathrm{B}(K_n) \xrightarrow{\mathbb{Z}_2} S^{n-2}$

$n - 2 \geq \operatorname{ind}(\mathrm{B}(G))$ $\square$

Theorem (Lovász). $\chi(G) \geq \operatorname{ind}(\mathrm{B}(G)) + 2$

- Useful

  Kneser's Conjecture

  $\ldots$

- But sometimes NOT

  $\square \not\subseteq G \ldots$

Topological lower bounds

J. Matoušek, G. M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy*, preprint, 2004, to appear in Jahresbericht der DMV (arXiv:math.CO/0208072)

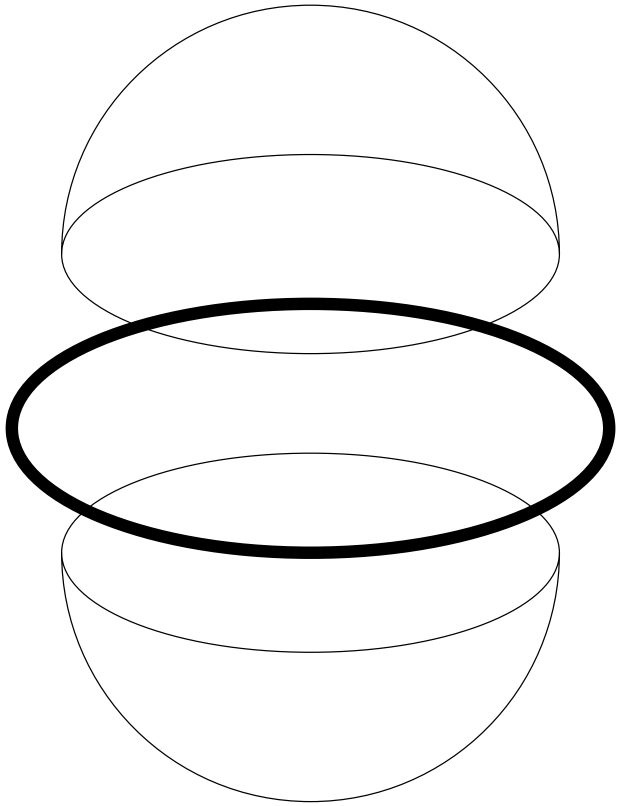
Lovász bound $\quad\quad \chi(G) \geq \operatorname{ind}(\mathrm{B}(G)) + 2$

Sarkaria bound $\quad\quad \chi(G) \geq \operatorname{ind}(\operatorname{susp}(\mathrm{B}(G))) + 1$

Bárány bound $\quad\quad$ Dol'nikov,Kříž bound

$$\operatorname{ind}(X) \leq \operatorname{ind}(\operatorname{susp}(X)) \leq \operatorname{ind}(X) + 1$$

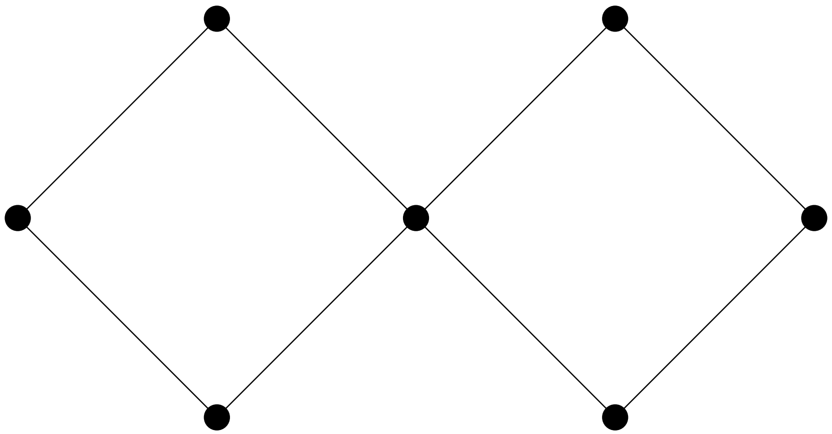
$$\mathrm{ind}\,(X) = \mathrm{ind}\,(\mathrm{susp}(X))$$



$h\colon S^3 \to S^2$ be the Hopf map

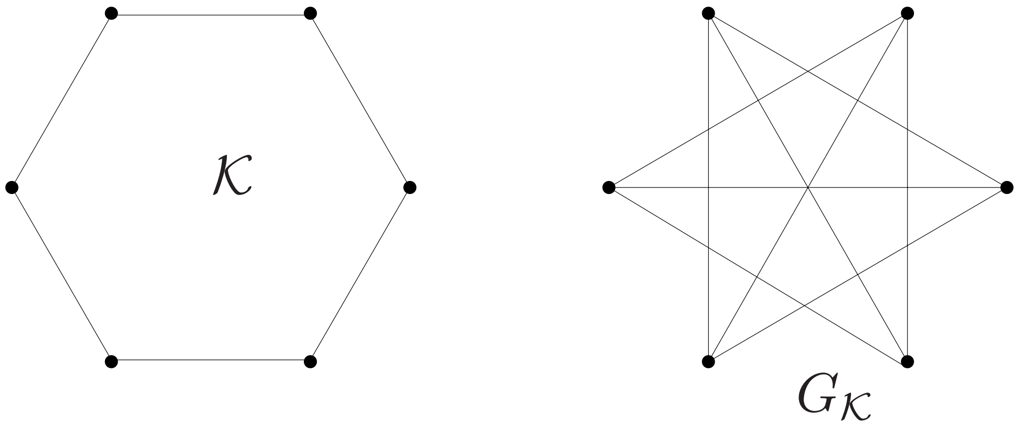
$$X_{2h} := S^2 \underset{2h}{\cup} B^4 \underset{-2h}{\cup} B^4.$$

Is it a box complex/neighborhood complex?

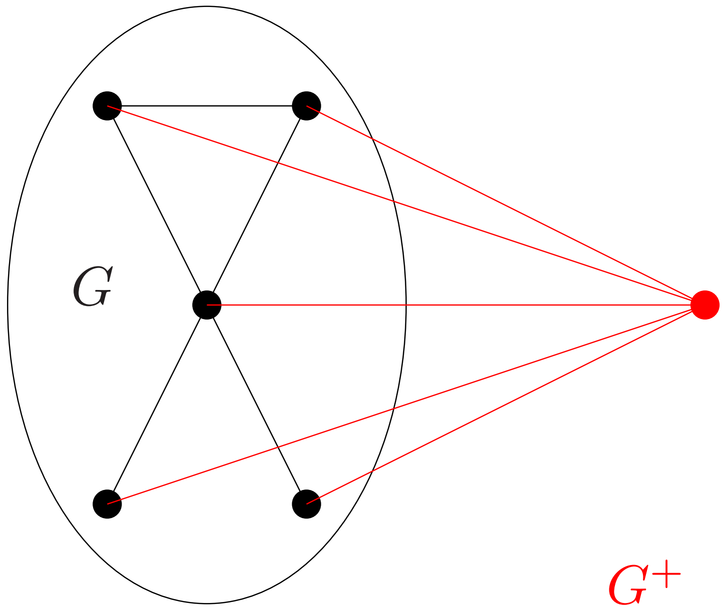


Theorem. *Given a free $\mathbb{Z}_2$-simplicial complex $(\mathcal{K}, \nu)$, there is a graph G such that its box complex $\mathrm{B}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the given complex.*

Construction



How good is $\chi(G) \geq \mathrm{ind}\,(\mathrm{B}(G)) + 2$?



$\mathrm{ind}\,(X) = \mathrm{ind}\,(\mathrm{susp}^k(X))$

$G := G_{\mathrm{sd}(X)}$

$\mathrm{B}(G^+) \simeq \mathrm{susp}(\mathrm{B}(G))$

$\chi\big(G^{k+}\big) \geq \mathrm{ind}\,(\mathrm{B}(G^{k+})) + 2 = \mathrm{ind}\,(\mathrm{susp}^k(X)) + 2 = \mathrm{ind}\,(X) + 2$